QCD based equation of state at finite density with a critical point from an alternative expansion scheme

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muses

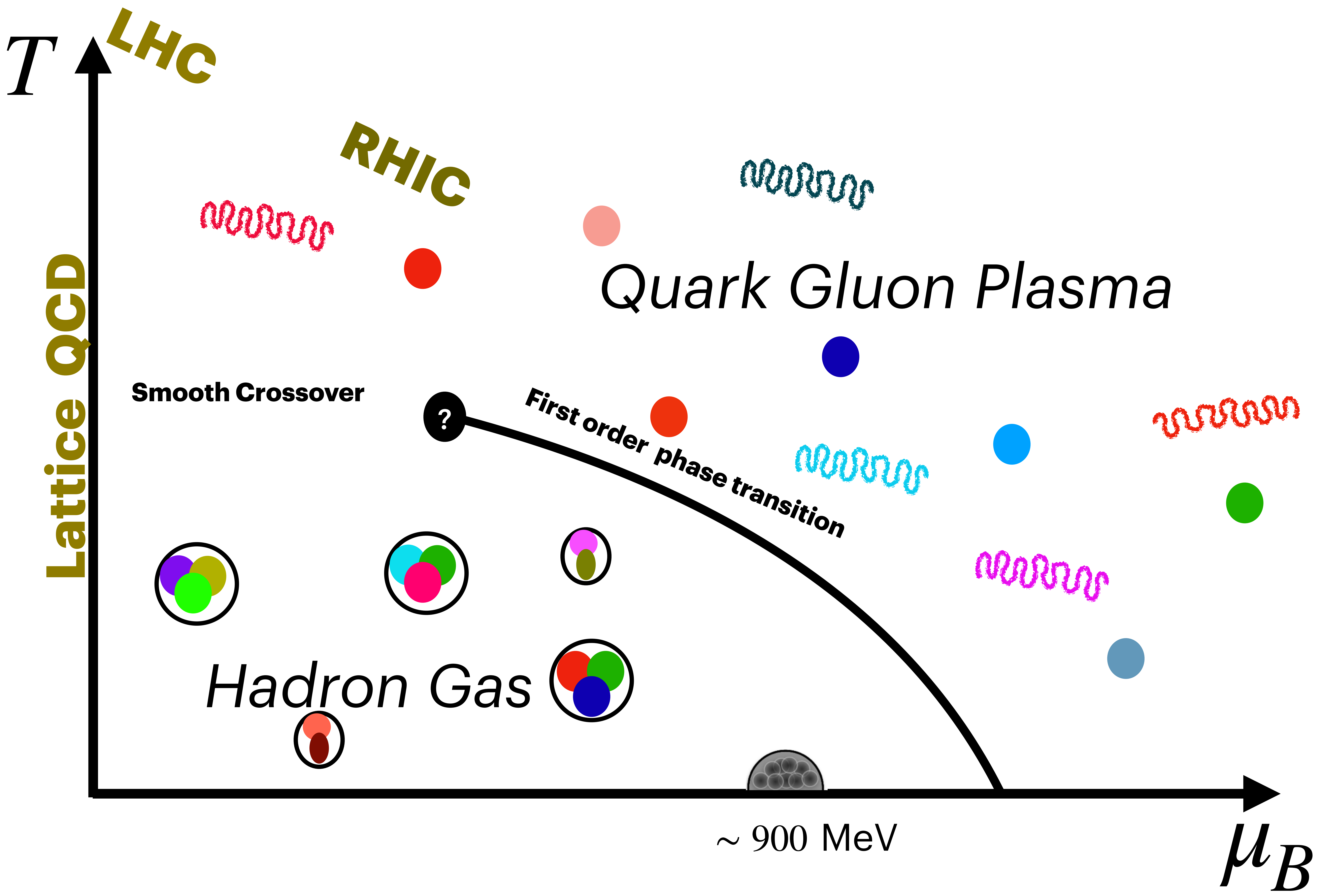
NP3M

NSF

NuSTEAM

Nuclear Science in Texas to Enhance and Advance Minorities

QCD Phase Diagram



Fermi-Sign Problem

- Lattice simulations at finite μ_B are challenging

Taylor: Lattice QCD results

Taylor Expansion around $\mu_B = 0$

$$\frac{P(T,\mu_B)}{T^4} = \sum_{n=0}^{\infty} \frac{1}{n!}\chi_n(T,\mu_B=0)\left(\frac{\mu_B}{T}\right)^n$$

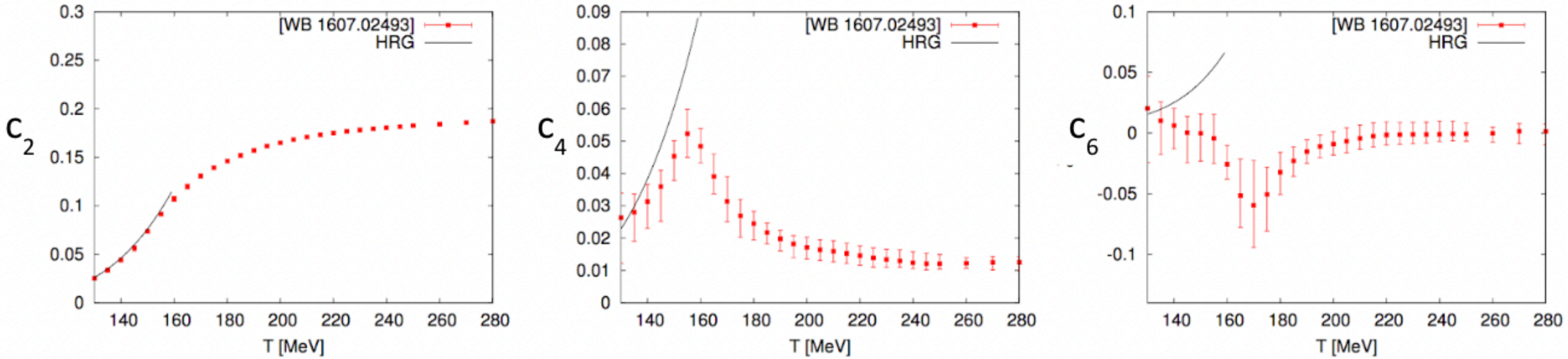
[Borsanyi, S. et al High Energy Physics.9(8), 1-16.(2012)]
[Bazavov, A et al PhysRevD.95, 054504 (2017)]

$$c_n(T) = \frac{\chi_n^B(T,\mu_B=0)}{n!} = \frac{1}{n!}\left(\frac{\partial}{\partial(\mu_B/T)}\right)^n (P/T^4)\Big|_{\mu_B=0}$$

Limitations

- Currently limited to $\frac{\mu_B}{T} \leq 2-3$ despite great computational power
- Adding one more higher-order term does not help in convergence
- Taylor expansion is carried out at T= constant and doesn't cope well with μ_B-dependent transition temperature

[Bollweg, D. et al PhysRevD. 108(1), 105, 074511 (2022)]
[Bollweg, D. et al PhysRevD. 108(1), 014510.(2023)]

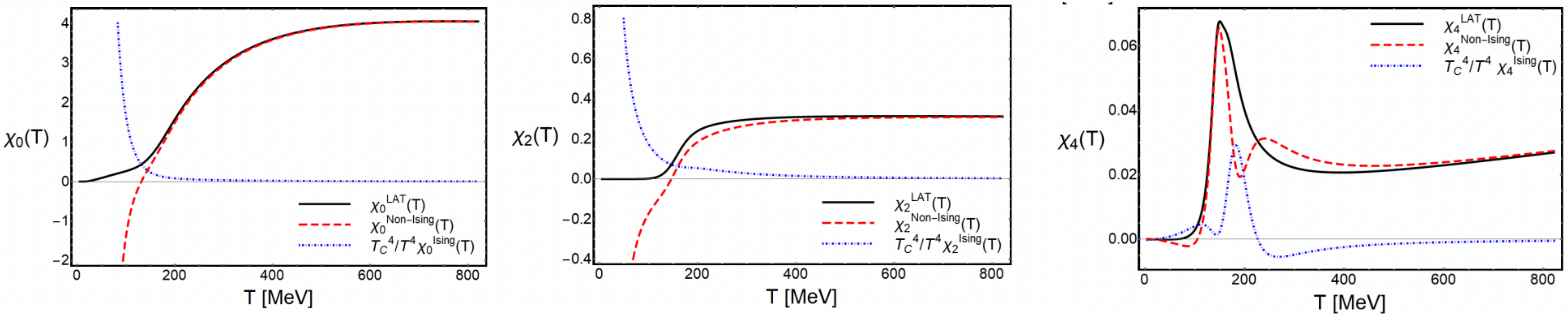


[WB Lattice QCD Collaboration]

Taylor: Lattice QCD results with a critical point

$$P(T,\mu_B) = T^4 \sum_{n=0}^{2} c_{2n}^{Non-Ising}(T)\left(\frac{\mu_B}{T}\right)^{2n} + T_C^4 P_{symm}^{Ising}(T,\mu_B)$$

$$\chi_n^{Lat}(T) = \chi_n^{Non-Ising}(T) + \frac{T_C^4}{T^4}\chi_n^{Ising}(T)$$



[P Parotto, et al PhysRevC. 108(1), 101.034901(2020)]

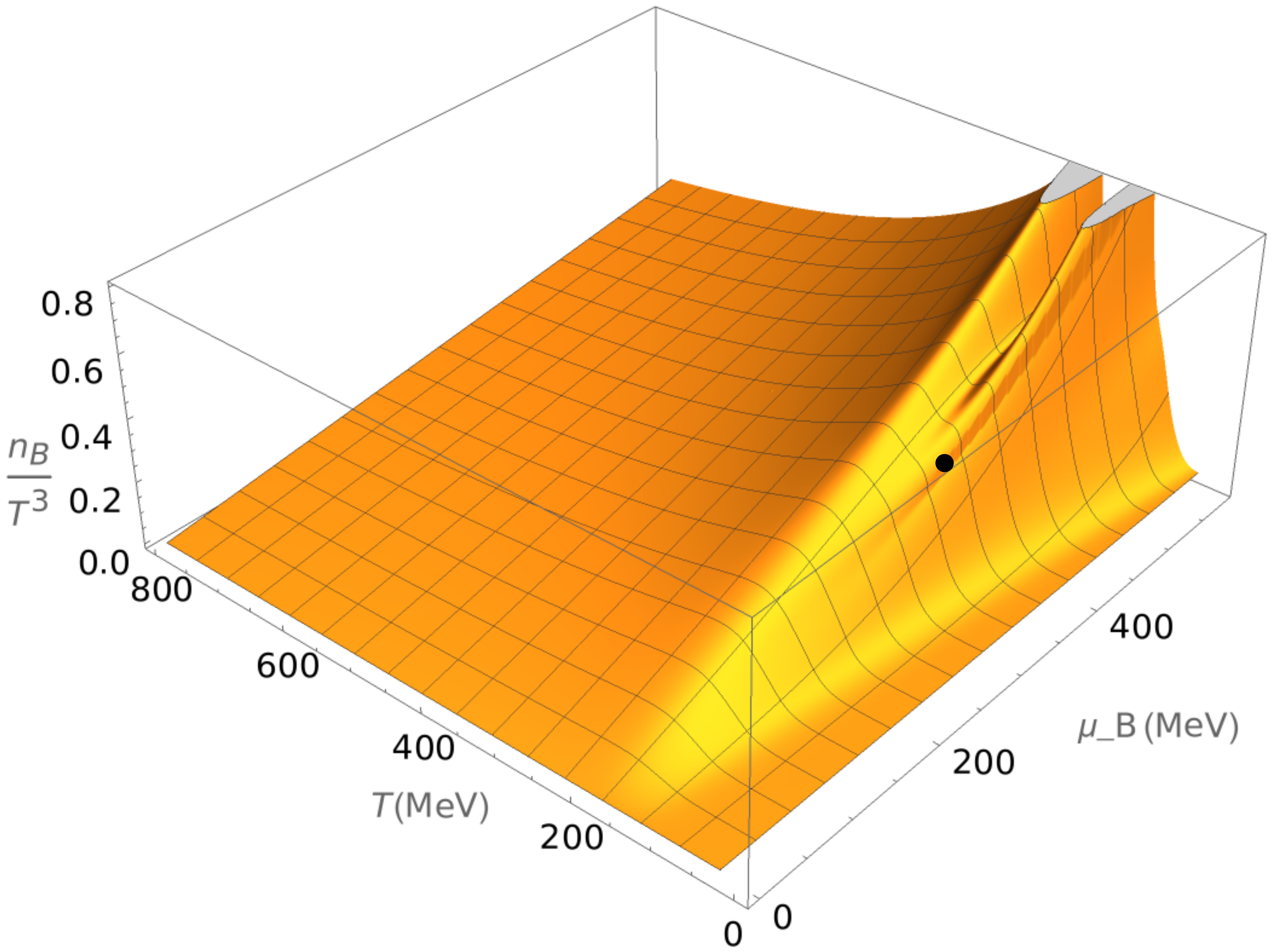
Taylor: Lattice QCD results with a critical point

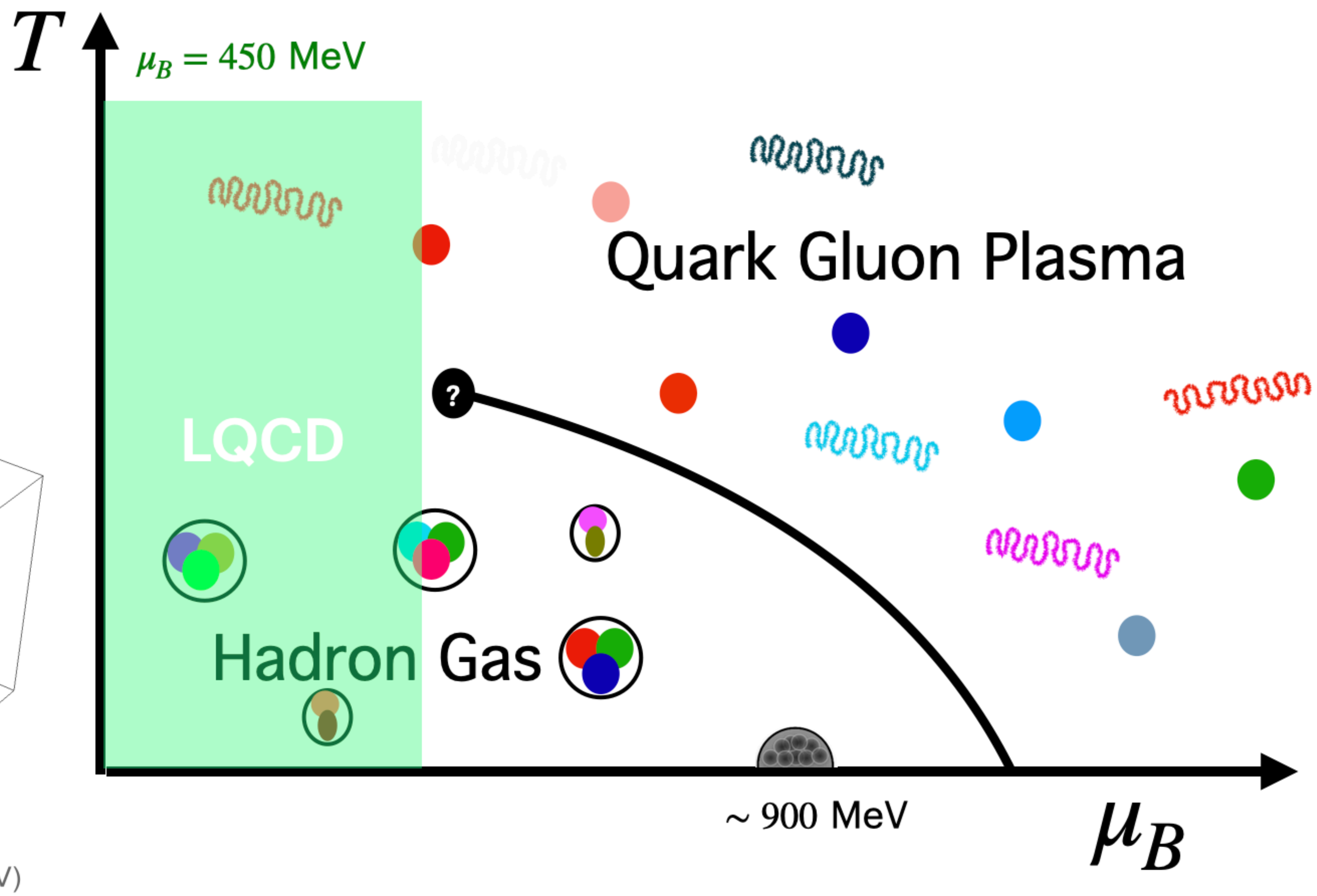
BEST COLLABORATION

$$\frac{n_B(T,\mu_B)}{T^3} = \frac{\partial(P/T^4)}{\partial(\mu_B/T)}$$

Critical Point at $\mu_{BC} = 350\ [MeV]$

Baryon density





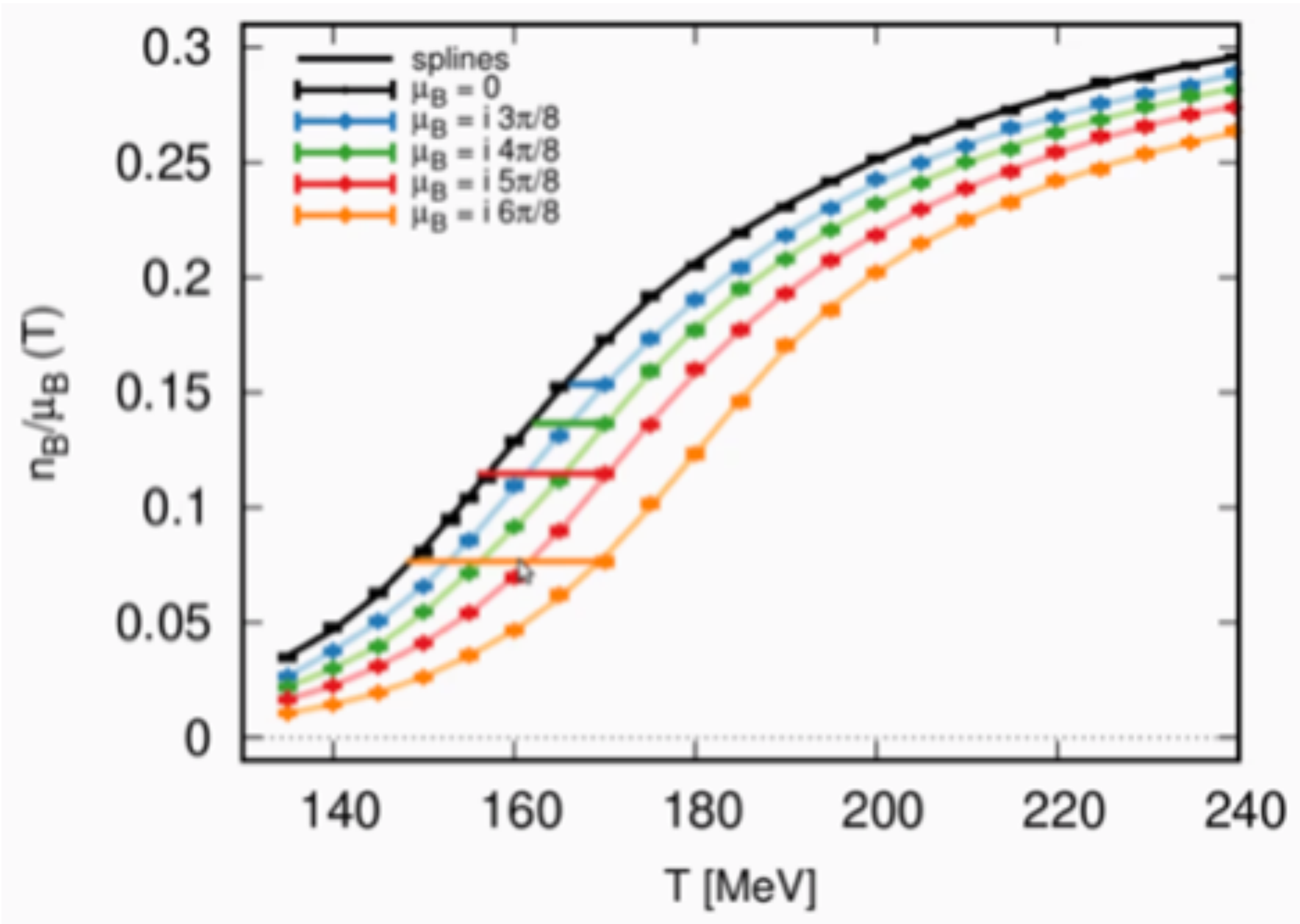
- Thermodynamic variables at higher $\mu_B \geq 450 MeV$ show **unphysical behavior**

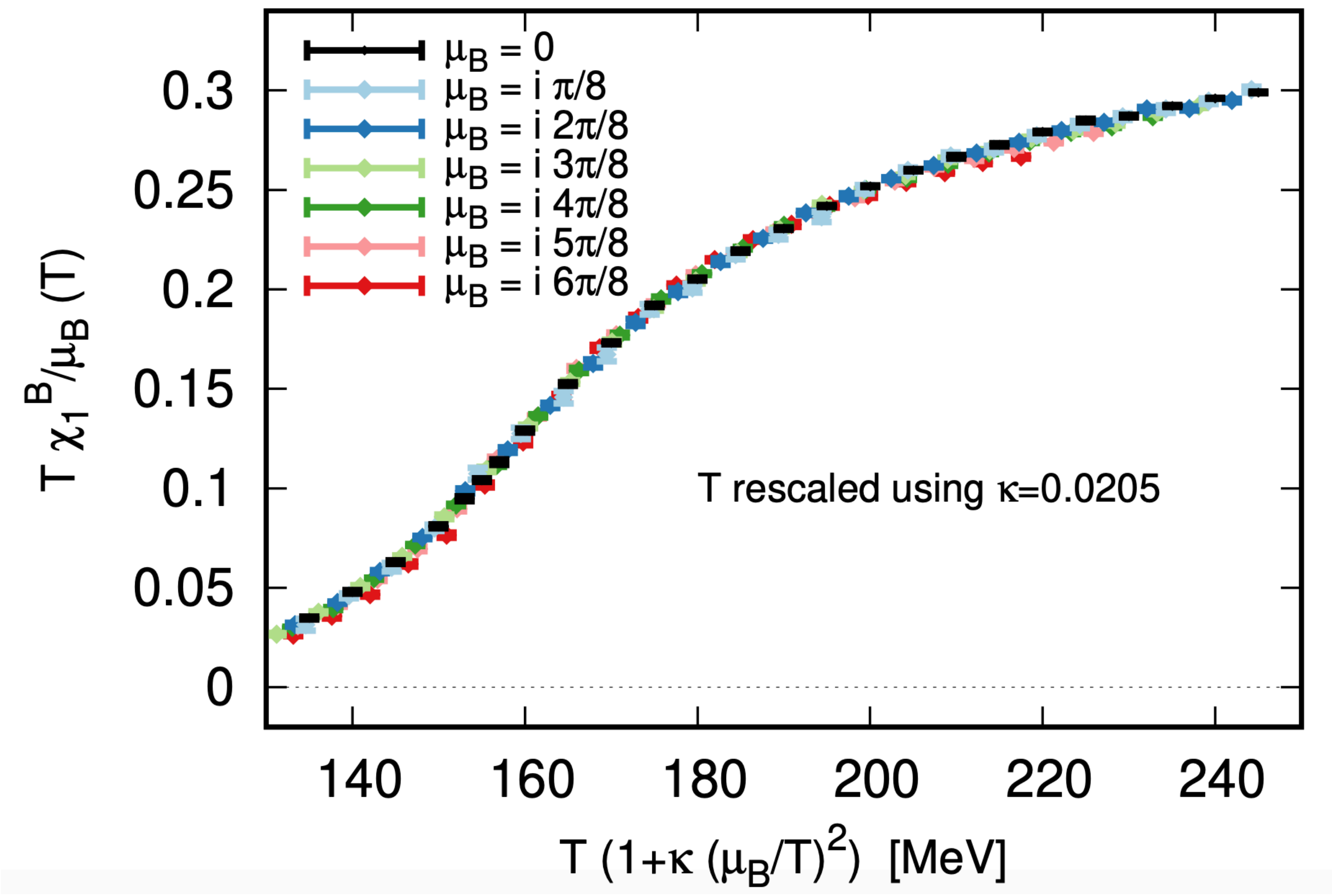
[P Parotto, et al PhysRevC. 108(1), 101.034901(2020)]
[Karthein, J, et al arXiv:2110.00622.(2021)]

Part 2: Lattice EoS: Alternative Expansion Scheme

Alternative Expansion scheme

Simulating at Imaginary μ_B





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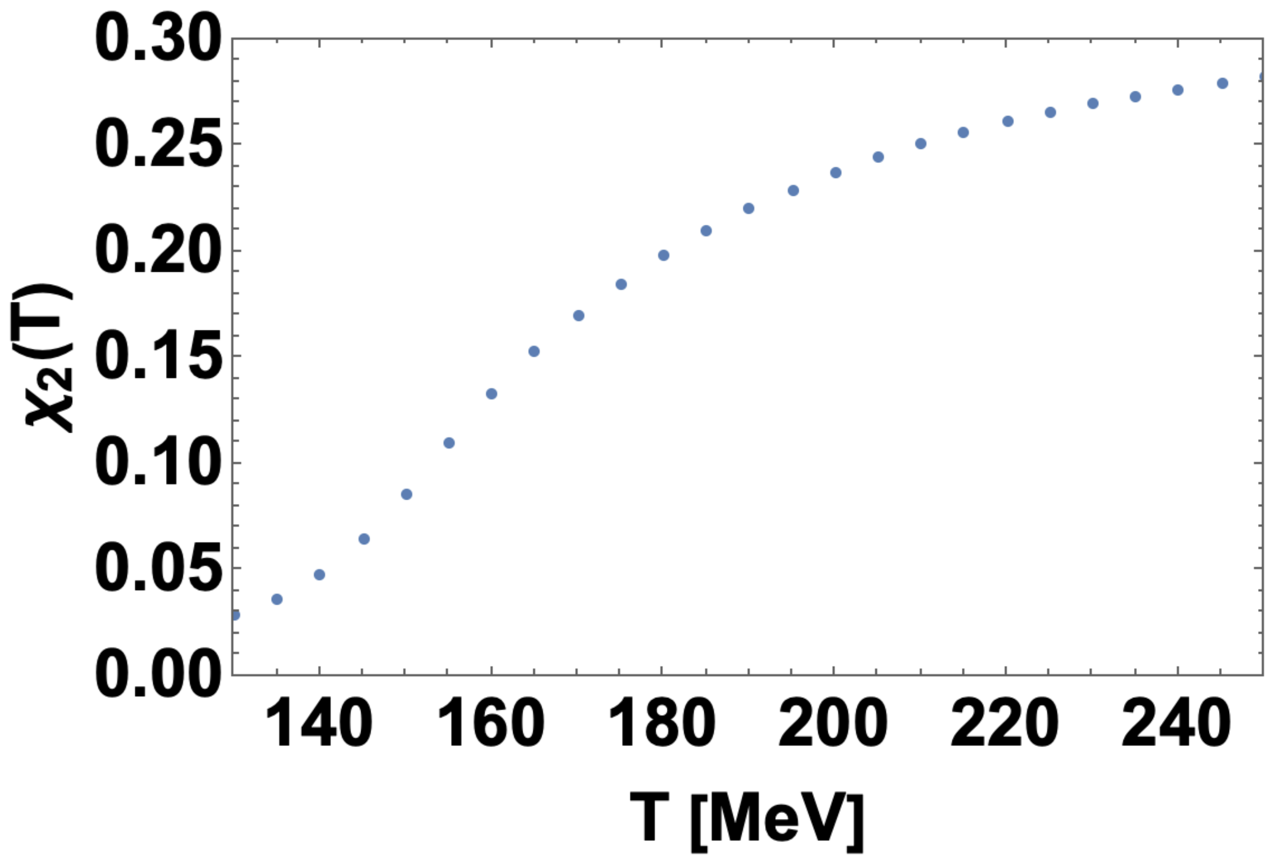


Borsányi, S et al PRL. 108(1), 101.034901(2021)]

$$T\frac{\chi_1^B(T,\mu_B)}{\mu_B} = \chi_2^B(T',0)$$

$$\chi_n^B(T,\mu_B) = \frac{1}{n!}\left(\frac{\partial}{\partial(\mu_B/T)}\right)^n (P/T^4)$$

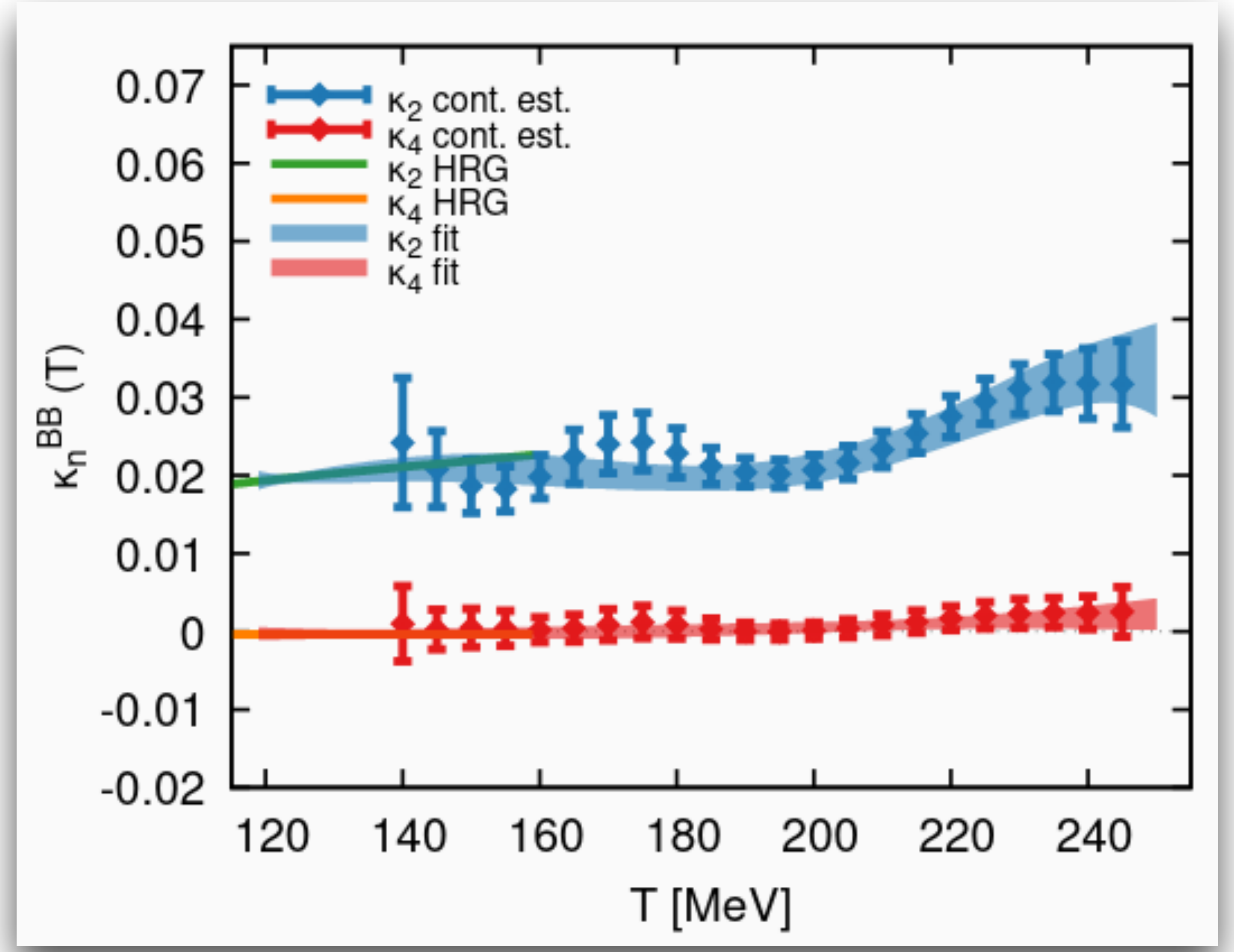
$$T'[T,\mu_B] = T\left[1 + \kappa_2^{BB}(T)\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^{BB}(T)\left(\frac{\mu_B}{T}\right)^4 + \mathcal{O}\left(\frac{\mu_B}{T}\right)^6\right]$$

- **μ_B dependence is captured in T-rescaling.**
- **Trusted up to** $\frac{\mu_B}{T} = 3.5$

Alternative Expansion scheme

Comparing **Taylor expansion** and **Alternative expansion**

- $\kappa_2^{BB}(T) = \frac{1}{6T} \frac{\chi_4^B(T)}{\left(\frac{\partial \chi_2^B(T)}{\partial T}\right)}$
- $\kappa_4^{BB}(T) = \frac{1}{360T\chi_2'^B(T)^3}\left(3\chi_2'^{B2}\chi_6^B(T) - 5\chi_2^B(T)''\chi_4^B(T)^2\right)$



Borsányi, S et al PRL. 108(1), 101.034901(2021)]

Pros

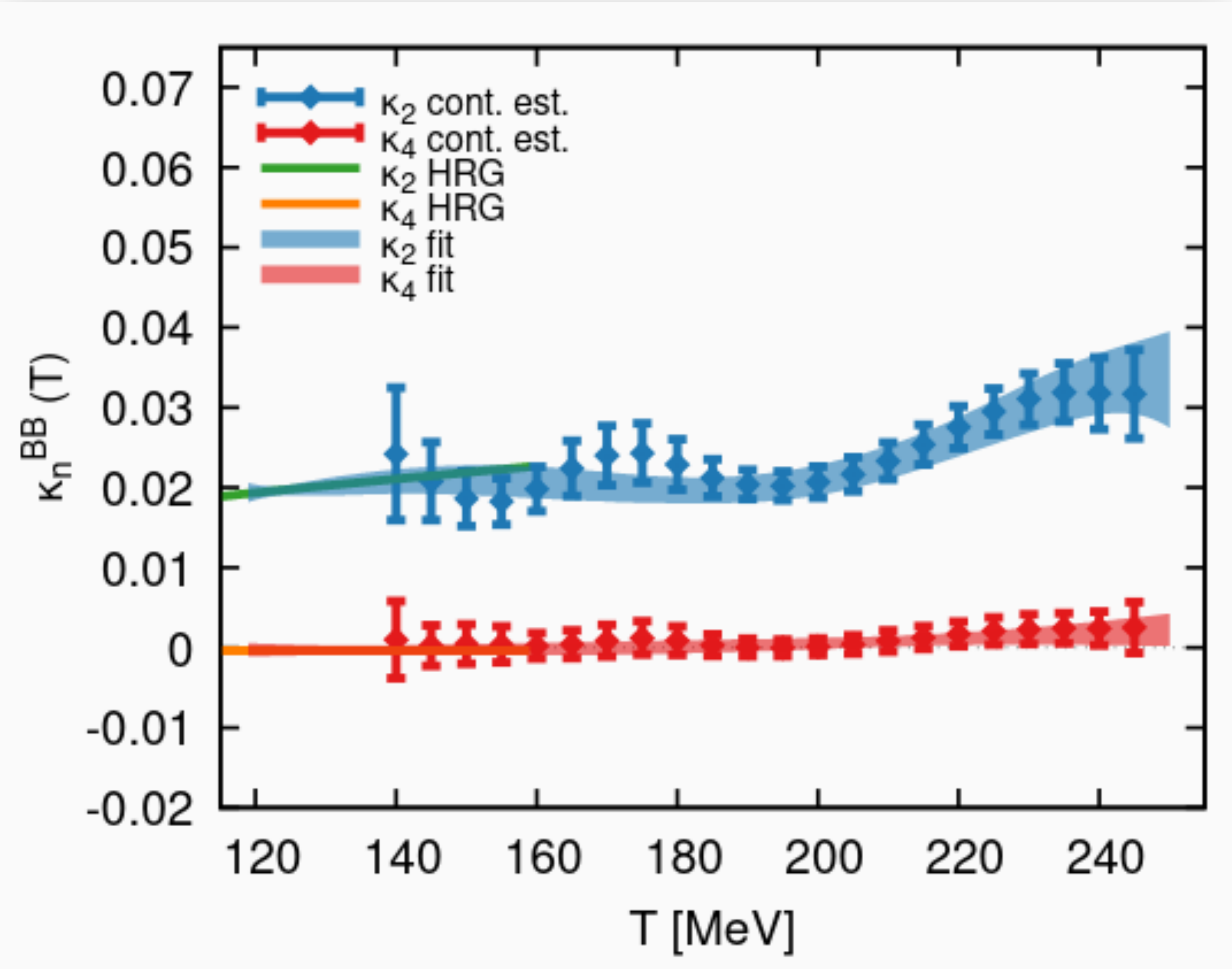
- $\kappa_2(T)$ **is fairly constant over a large T-Range**
- **There is a separation of scale between** $\kappa_2(T)$ **and** $\kappa_4(T)$
- $\kappa_4(T)$ **is almost zero** $\rightarrow$ **faster convergence**
- **A good agreement with HRG results at Low Temperature**

Alternative Expansion scheme

Comparing **Taylor expansion** and **Alternative expansion**

- $\kappa_2^{BB}(T) = \dfrac{1}{6T}\dfrac{\chi_4^B(T)}{\left(\dfrac{\partial \chi_2^B(T)}{\partial T}\right)}$

- $\kappa_4^{BB}(T) = \dfrac{1}{360T\chi_2'^B(T)^3}\left(3\chi_2'^{B2}\chi_6^B(T) - 5\chi_2^B(T)''\chi_4^B(T)^2\right)$



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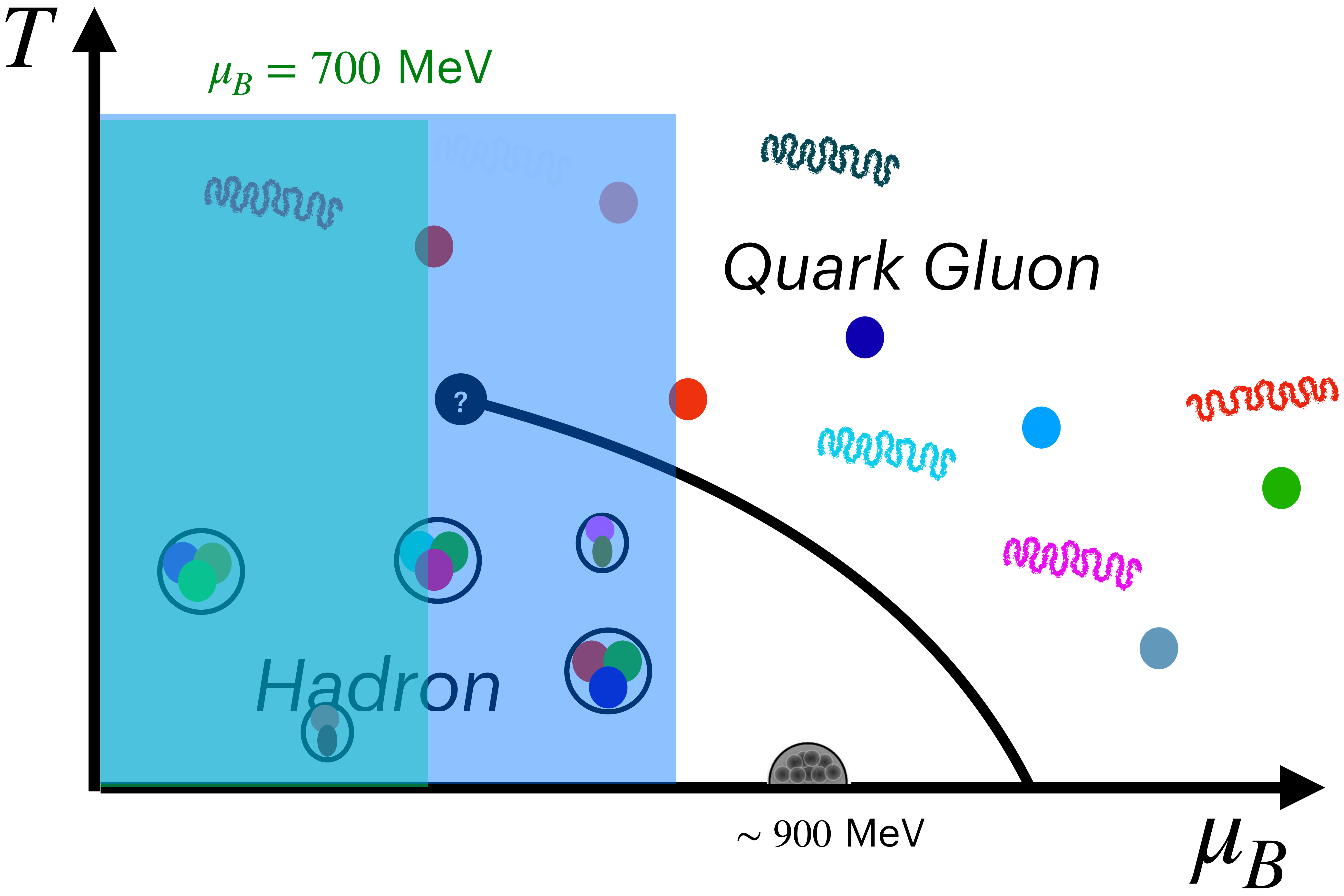


Part 3: 3D-Ising: Introducing Critical Point

EoS with a Critical point

Strategy

Scale 3D-Ising model EoS → **Map Ising to QCD** → **Merge Ising with Lattice** → **Estimate the critical Contribution**

Scale 3D-Ising Model EoS

Close to the critical point, we define a parametrization for Magnetization M, Magnetic field h, and reduced temperature

QCD Critical point is in the 3D-Ising model Universality class

$$M = M_0 R^{\beta} \theta$$

$$h = h_0 R^{\beta\delta} \tilde{h}(\theta)$$

$$r = R(1 - \theta^2)$$

$$(R \geq 0,\ |\theta| \leq \theta_0)$$

$$(R, \theta) \longmapsto (r, h)$$

$$\alpha = 0.11$$

$$\delta \sim 4.8$$

$$\beta \sim 0.326$$

$$r = \frac{T - T_C}{T_C}$$

$h \rightarrow$ **External magnetic field**

$$P^{Ising}(R, \theta) = h_0 M_0 R^{2-\alpha} \left[\theta \tilde{h}(\theta) - g(\theta)\right]$$

$$g(\theta) = c_0 + c_1(1 - \theta^2) + c_2(1 - \theta^2)^2 + c_3(1 - \theta^2)^3$$

$$\tilde{h}(\theta) = (\theta + a\theta^3 + b\theta^5)$$

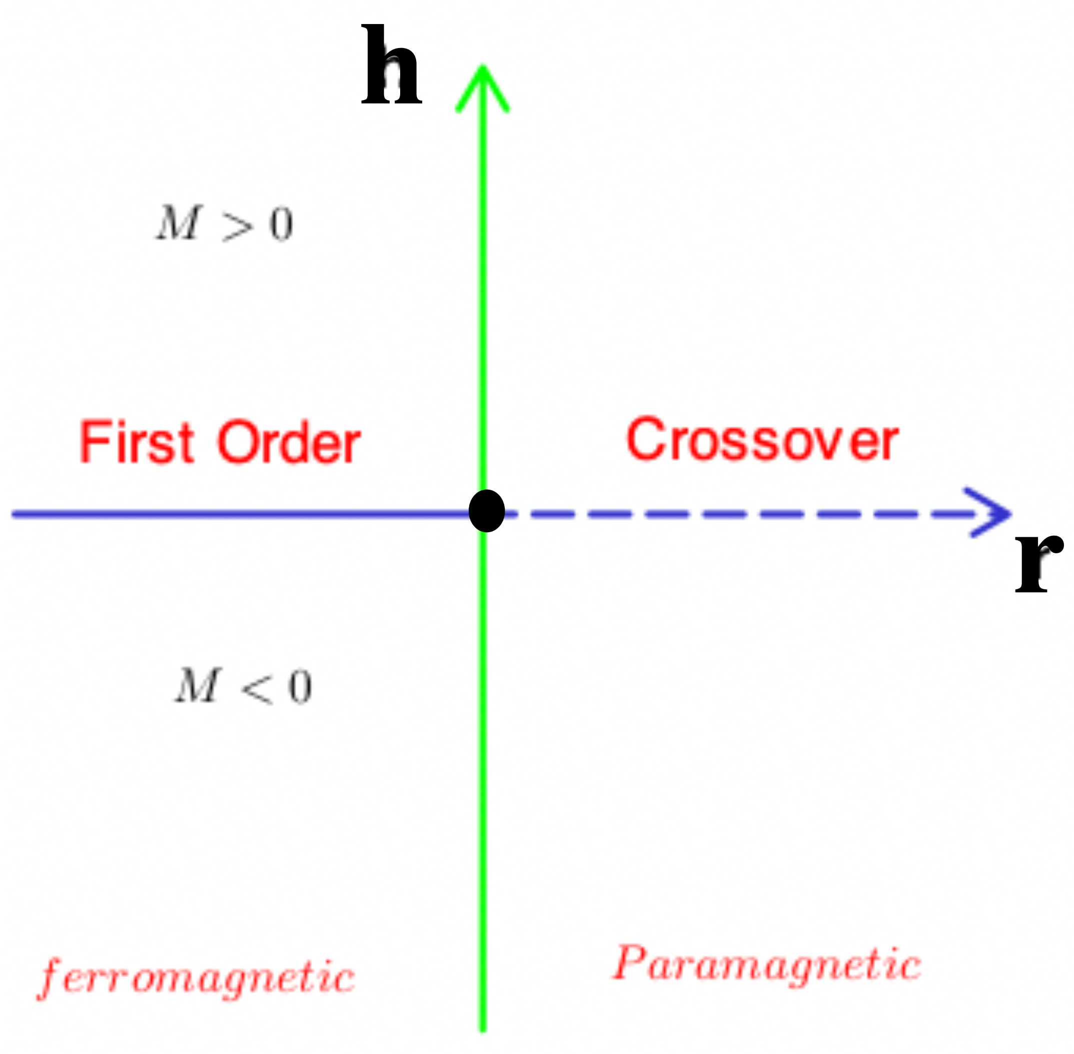
[Parotto et al PhysRevC.101.034901(2020)]

[Nonaka et al Physical Review C, 71(4), 044904.(2005)]

[Guida et al Nuclear Physics B, 489(3), 626-652.(1997)]

Introducing Critical Point

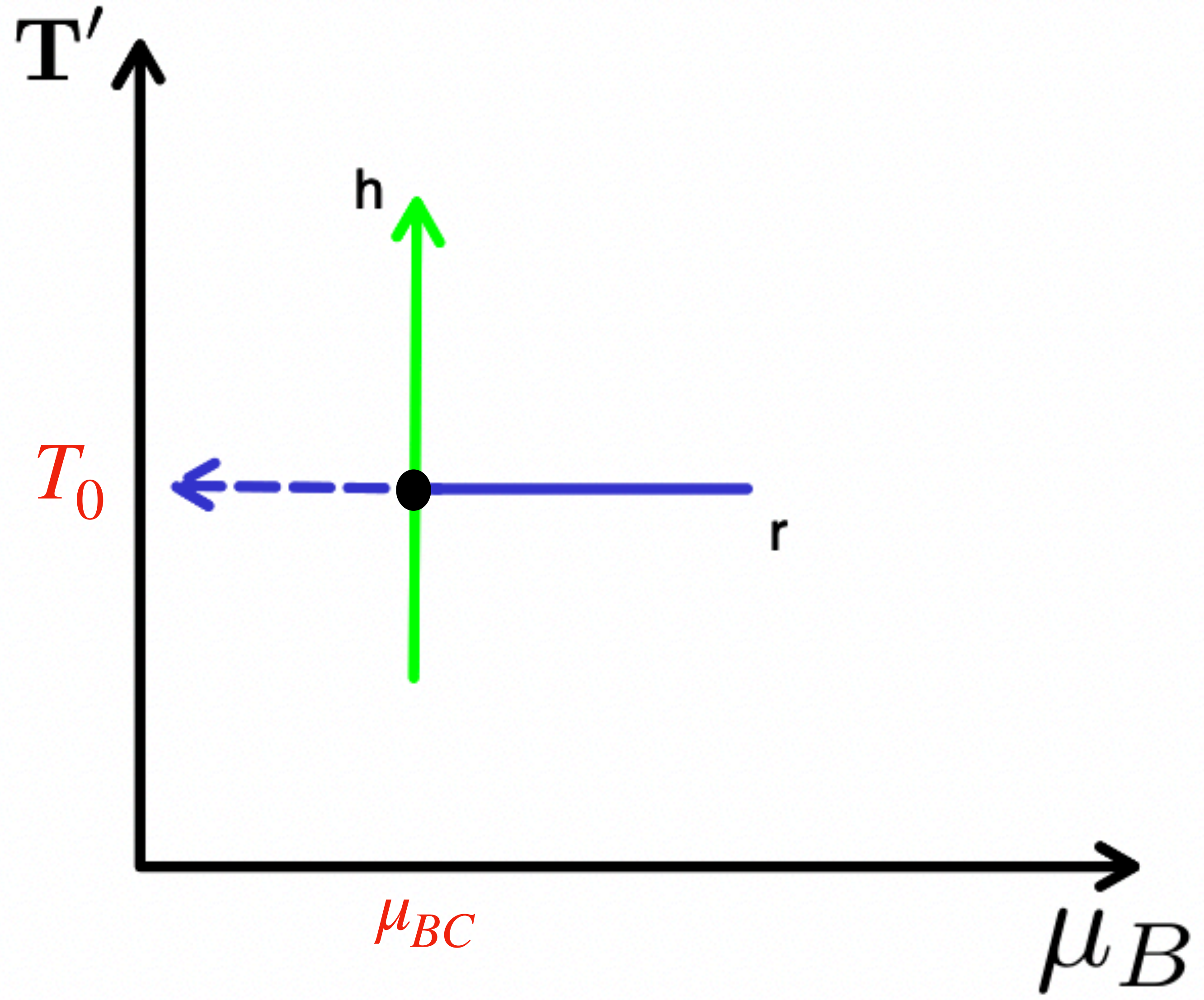
3D Ising to QCD Mapping



3D-Ising Co-ordinates

$$\frac{T' - T_0}{T_0} = w\, h \sin \alpha_{12}$$

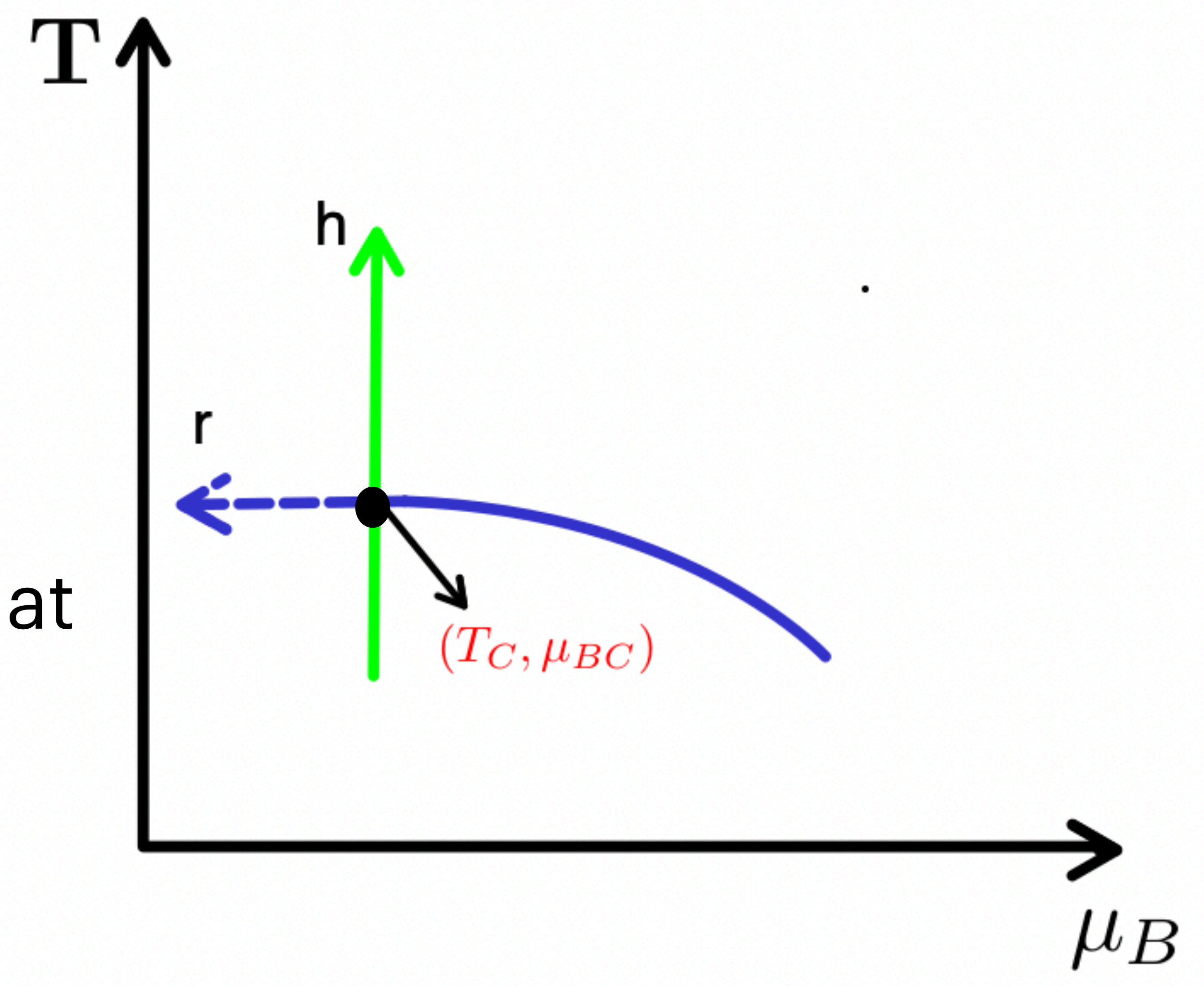
$$\frac{\mu_B^2 - \mu_{BC}^2}{T_0^2} = w(-r\rho + h \cos \alpha_{12})$$



(T', μ_B)-Co-ordinates

$$T' = T\left[1 + \left(\frac{\mu_B}{T}\right)^2 \kappa_2^{BB}(T) + \mathcal{O}\left(\frac{\mu_B}{T}\right)^4\right]$$

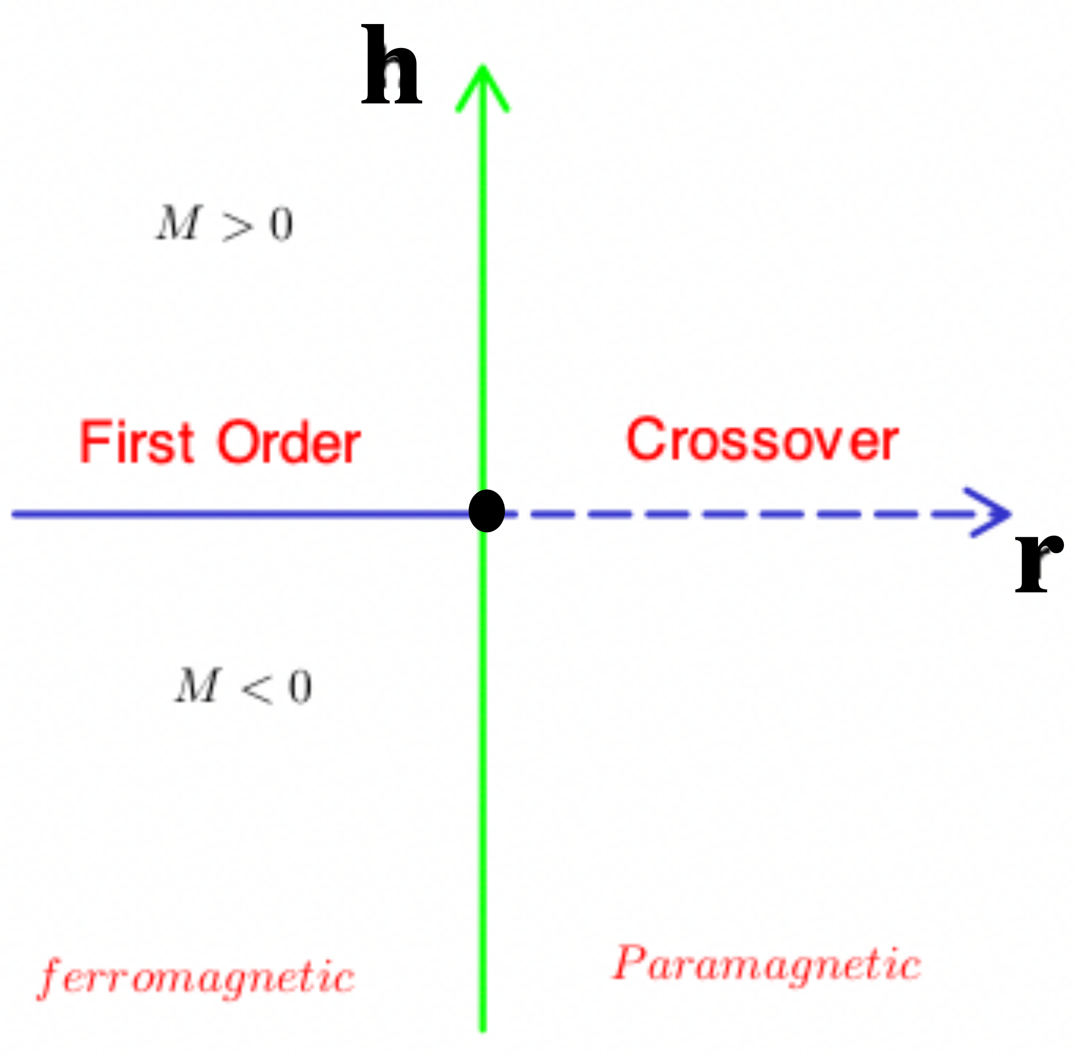
T_0 is the Pseudo Critical Temperature at $\mu_B = 0$



QCD - Co-ordinates

Introducing Critical Point

3D Ising to QCD Mapping



3D-Ising Co-ordinates

$$\frac{T' - T_0}{T_0} = w\, h \sin \alpha_{12}$$

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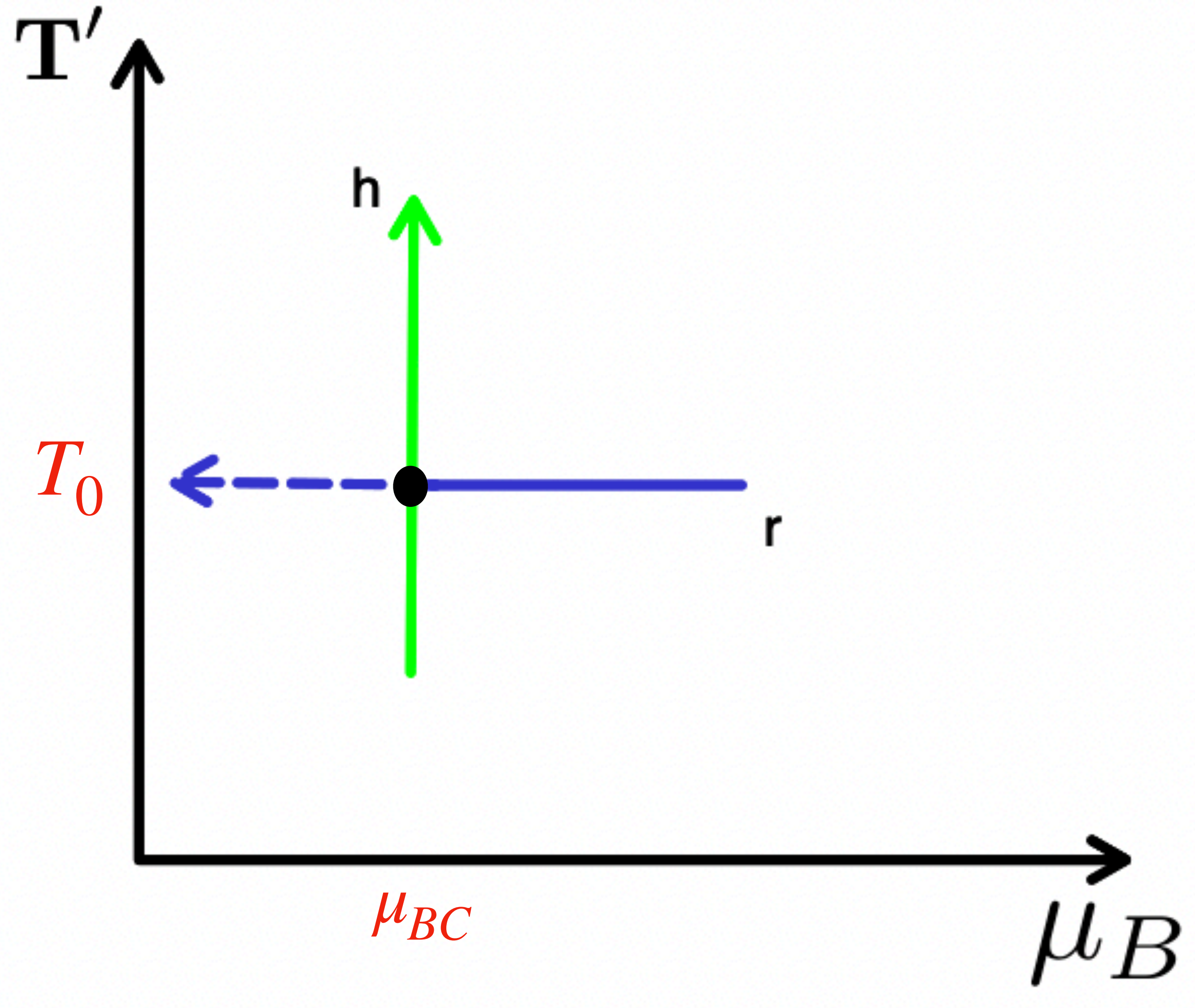
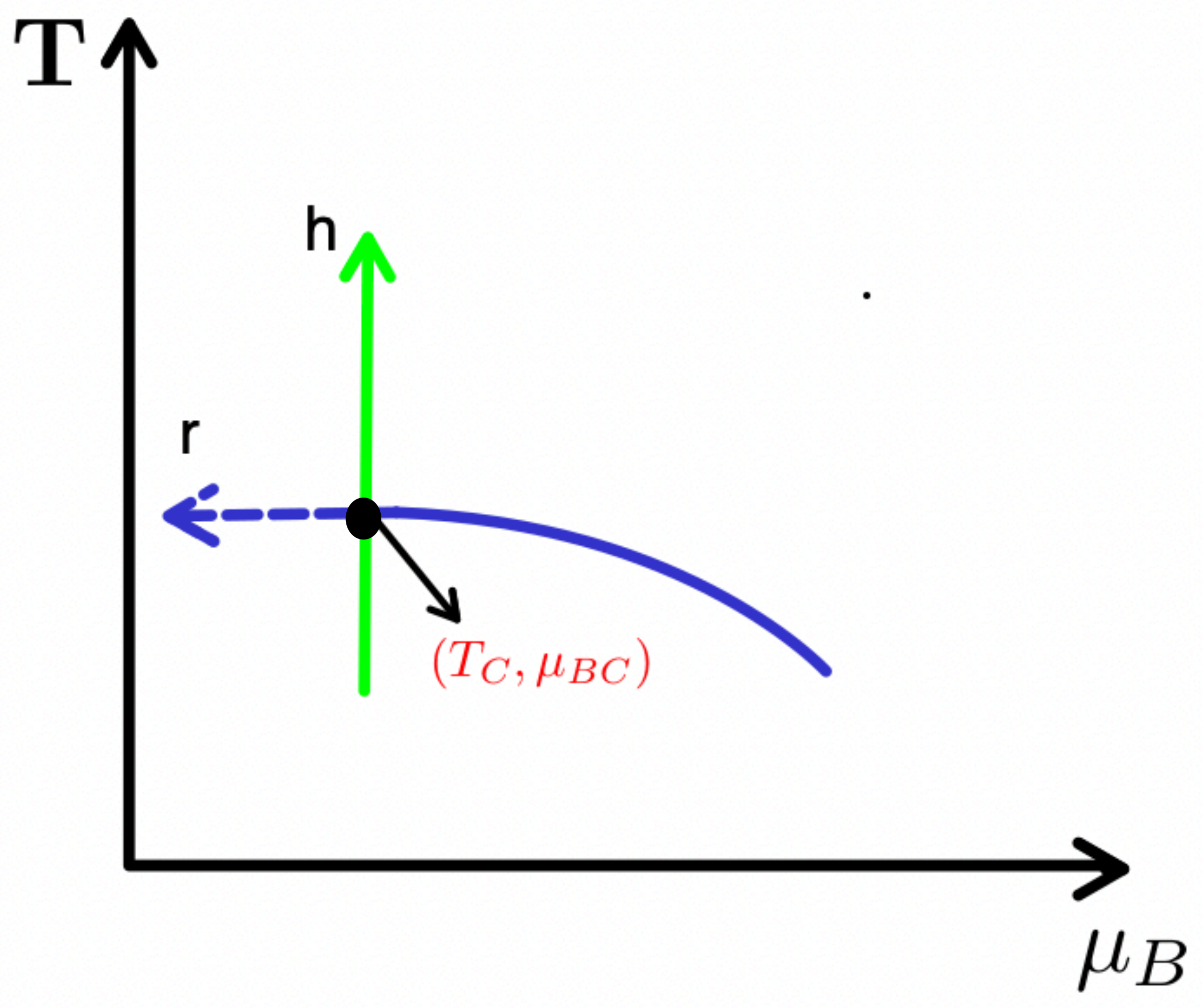


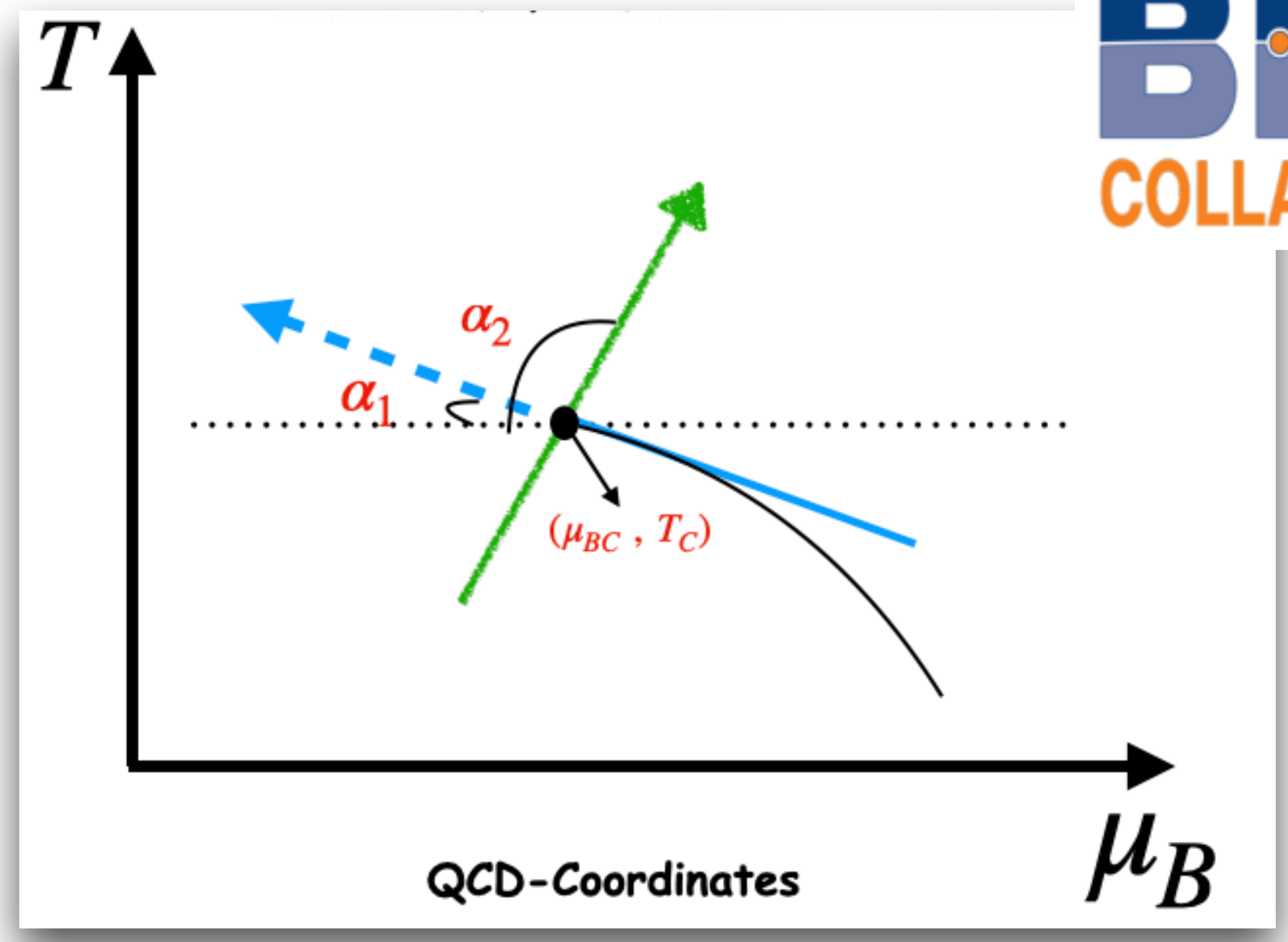
(T', μ_B)-Co-ordinates

T_0 is the Pseudo Critical Temperature

$\mu_B = 0$

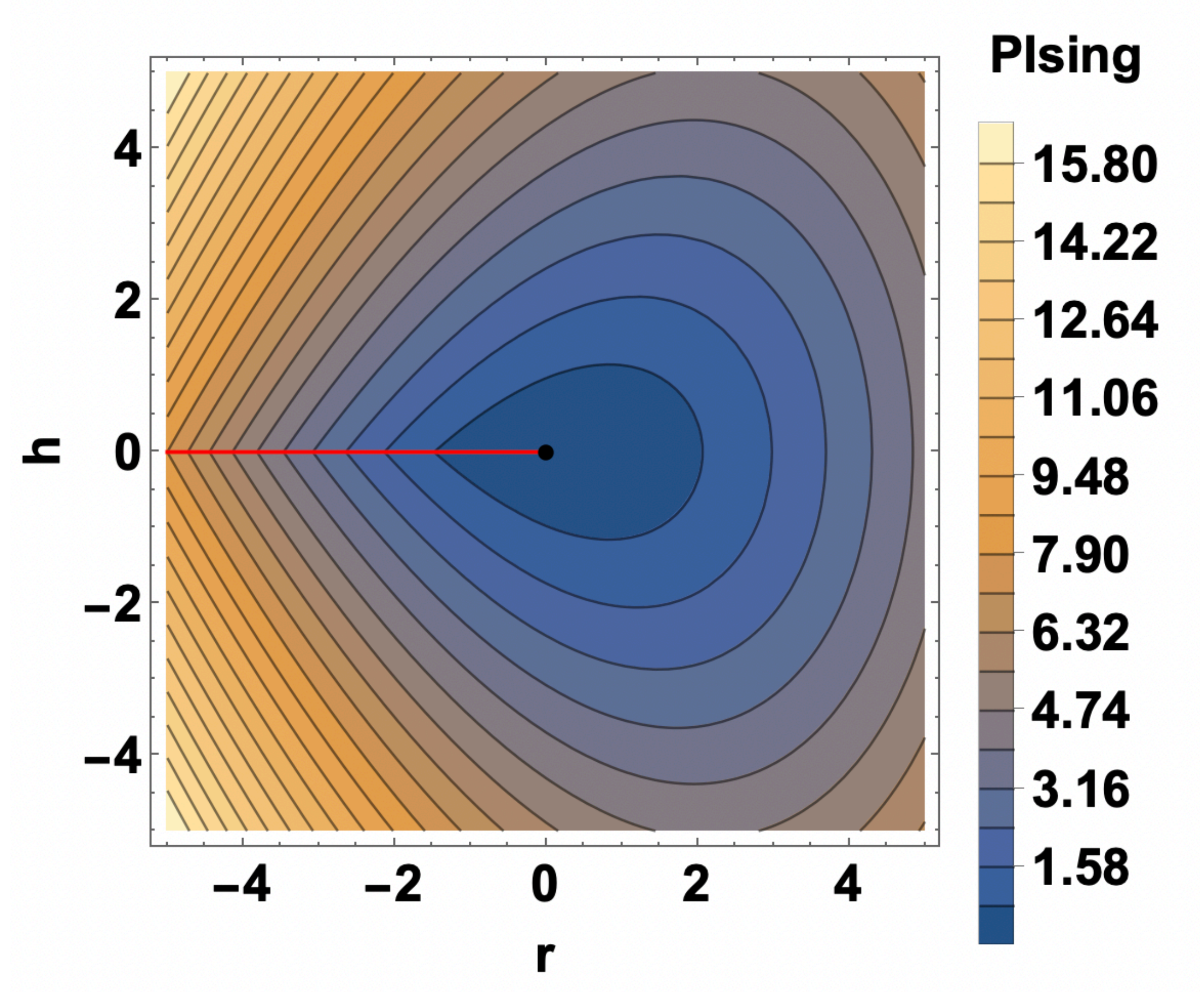


QCD - Co-ordinates



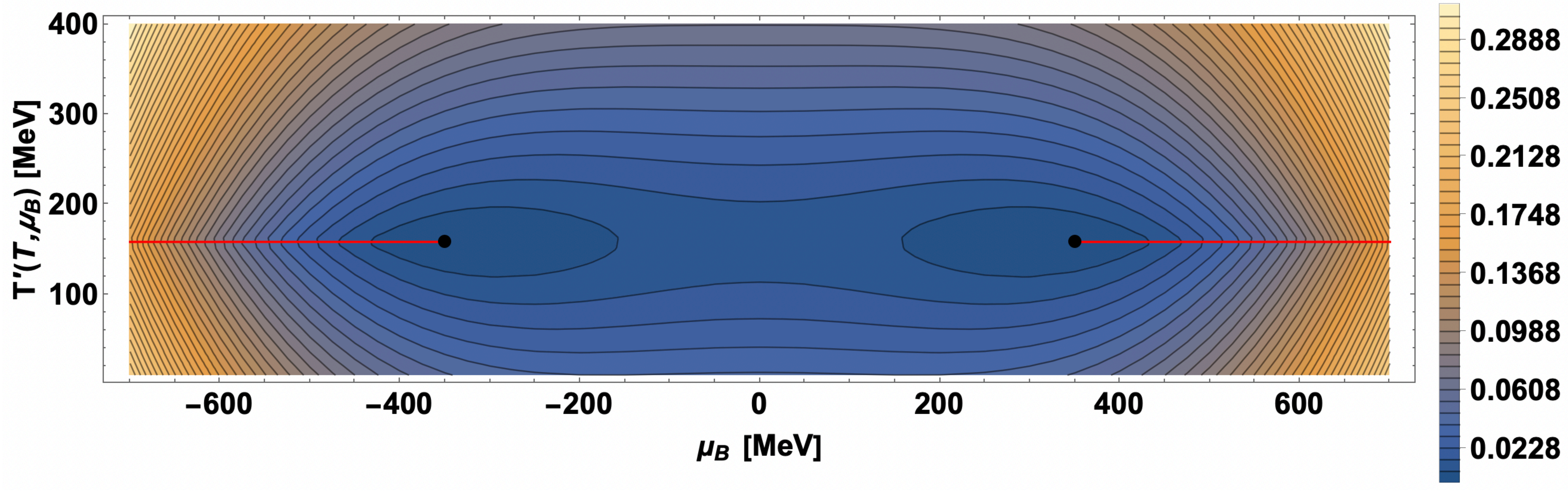
Introducing Critical Point

3D Ising to QCD Mapping

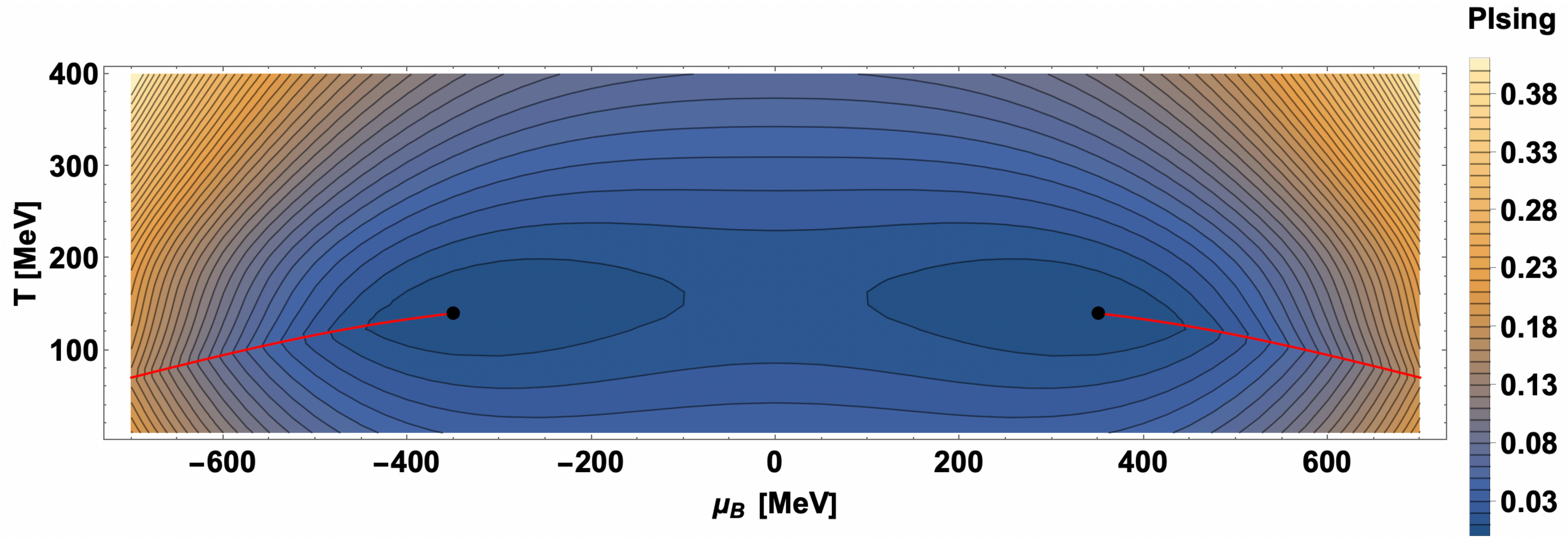


3D-Ising coordinates

Ising Pressure



T'-coordinates



QCD coordinates

Parameters

$w = 10,\ \rho = 0.5,\ T_0 = 158\ MeV$

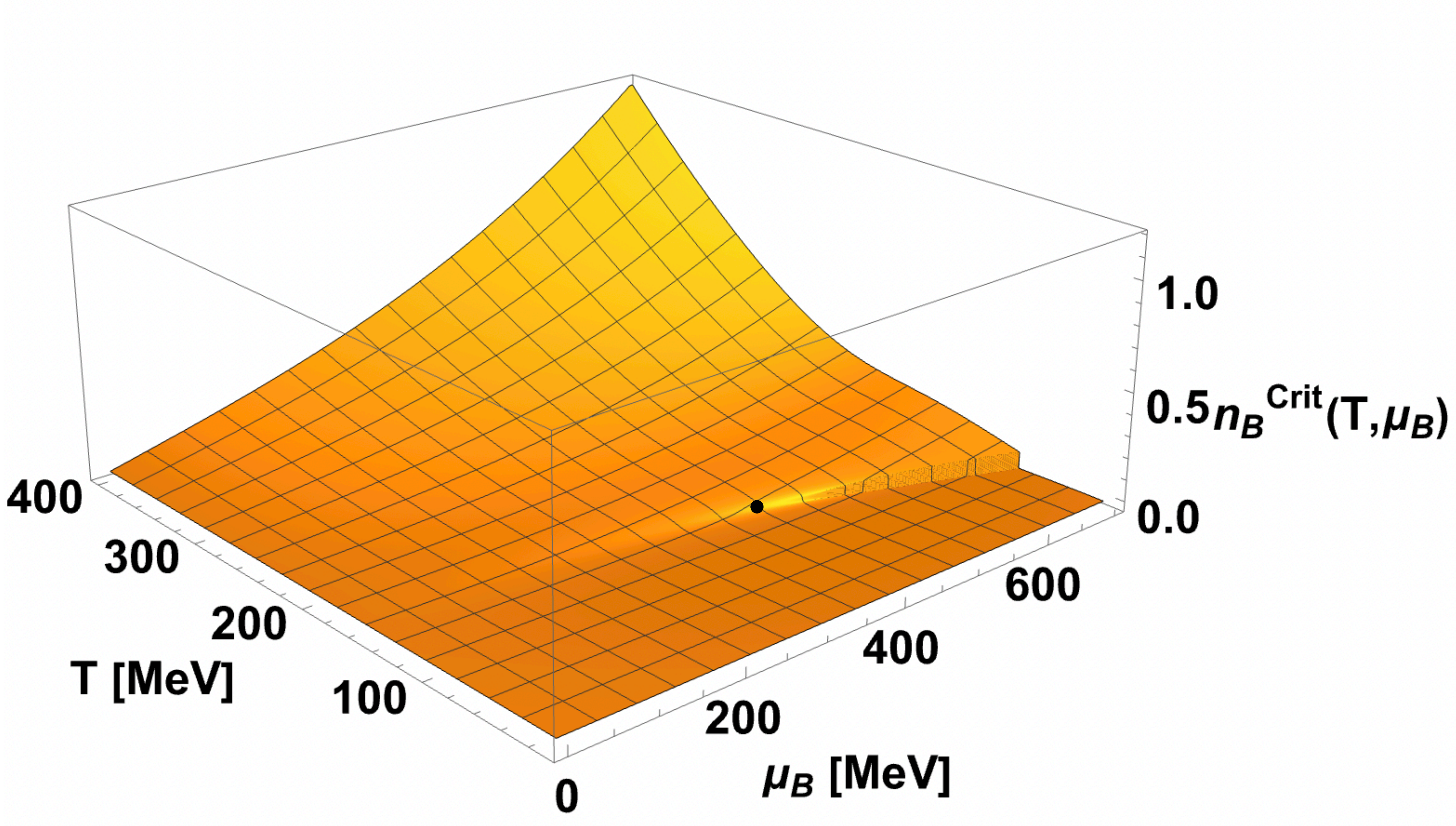
$\mu_{BC} = 350\ MeV,\ T_C = 140.07\ MeV$

$$T_C\left[1 + \kappa_2(T_C)\left(\frac{\mu_{BC}}{T_C}\right)^2\right] = T_0$$

Introducing Critical Point

Ising Baryon Density

$$\chi_1^{Ising}(T,\mu_B) = n_B^{Ising}(T,\mu_B) = \frac{\partial(P^{Ising}(T,\mu_B)/T^4)}{\partial(\mu_B/T)}$$

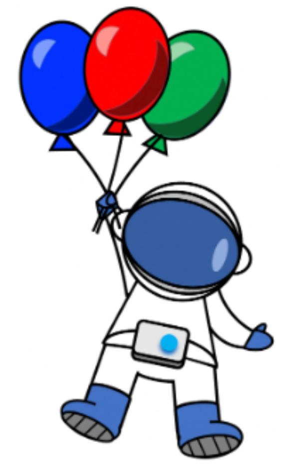


● Critical Point

$w = 2$, $\rho = 15$ & $\alpha_{12} = 90$

$\mu_{BC} = 500$ MeV

$T_C = 116.5$ MeV



Part 4: Merging Ising with Lattice

Merging Ising with Lattice

Full Baryon Density

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi^B_{2,Lattice}(T'_{full}(T,\mu_B),0)$$

$$T'[T,\mu_B] = T\left[1 + \kappa_2^{BB}(T)\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^{BB}(T)\left(\frac{\mu_B}{T}\right)^4 + \mathcal{O}\left(\frac{\mu_B}{T}\right)^6\right]$$

Merging Ising with Lattice

Full Baryon Density

$$\frac{n_B^{full}(T,\mu_B)}{T^3} = \left(\frac{\mu_B}{T}\right)\chi^B_{2,Lattice}(T'_{full}(T,\mu_B),0)$$

$$T'_{full}(T,\mu_B) = \underbrace{T'_{Lattice}(T,\mu_B)}_{\text{lowest order in } (\frac{\mu_B}{T})} + \underbrace{T'_{crit}(T,\mu_B) - Taylor[T'_{crit}(T,\mu_B)]}_{\text{higher orders in } (\frac{\mu_B}{T})}$$

Lattice Term **Ising Term**

Introducing a Critical Point

$$T'_{crit}(T,\mu_B) \approx T_0 + \left(\left.\frac{\partial\chi^B_{2,lattice}(T,0)}{\partial T}\right|_{T=T_0}\right)^{-1} \frac{n_B^{crit}(T,\mu_B)/T^3}{(\mu_B/T)} + \ldots$$

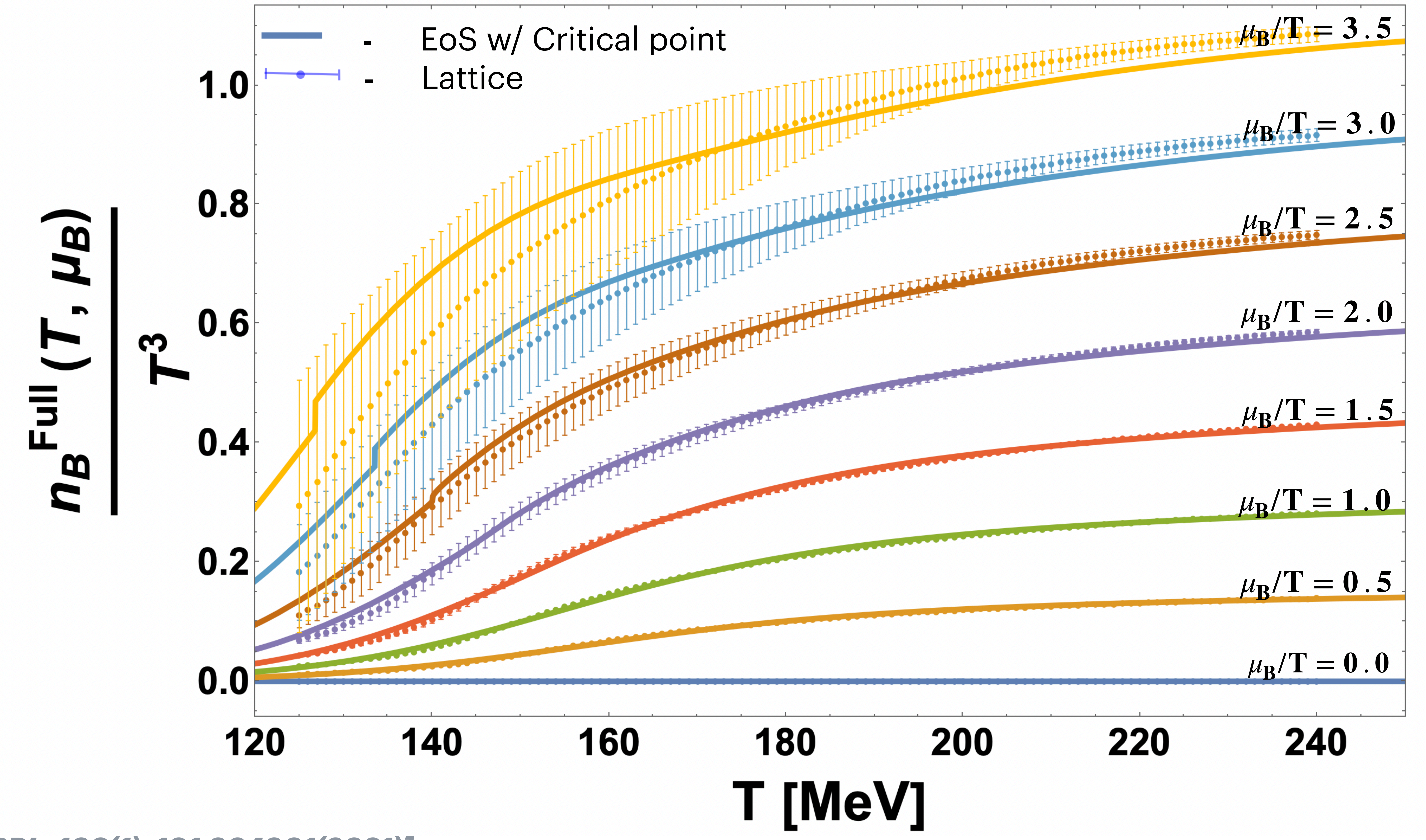
$$Taylor[T'_{crit}, n=2] \approx T_0 + \left(\left.\frac{\partial\chi^B_{2,lattice}(T,0)}{\partial T}\right|_{T=T_0}\right)^{-1} \left[\left.\frac{\partial(n_B^{crit}(T,\mu_B)/T^3)}{\partial(\mu_B/T)}\right|_{\mu_B/T=0} + \frac{1}{3!}\left.\frac{\partial^3(n_B^{crit}(T,\mu_B)/T^3)}{\partial(\mu_B/T)^3}\right|_{\mu_B/T=0}\left(\frac{\mu_B}{T}\right)^2 + \ldots\right]$$

$$\chi_1^{crit}(T,\mu_B) = \frac{n_B^{crit}(T,\mu_B)}{T^3} = \frac{\partial(P^{crit}(T,\mu_B)/T^4)}{\partial(\mu_B/T)}$$

Baryon density results

Full Baryon Density at a constant $\frac{\mu_B}{T}$ compared with Lattice

$\mu_B = 350$ [MeV], $\alpha_{12} = 90$, $\rho = 2$, $w = 2$



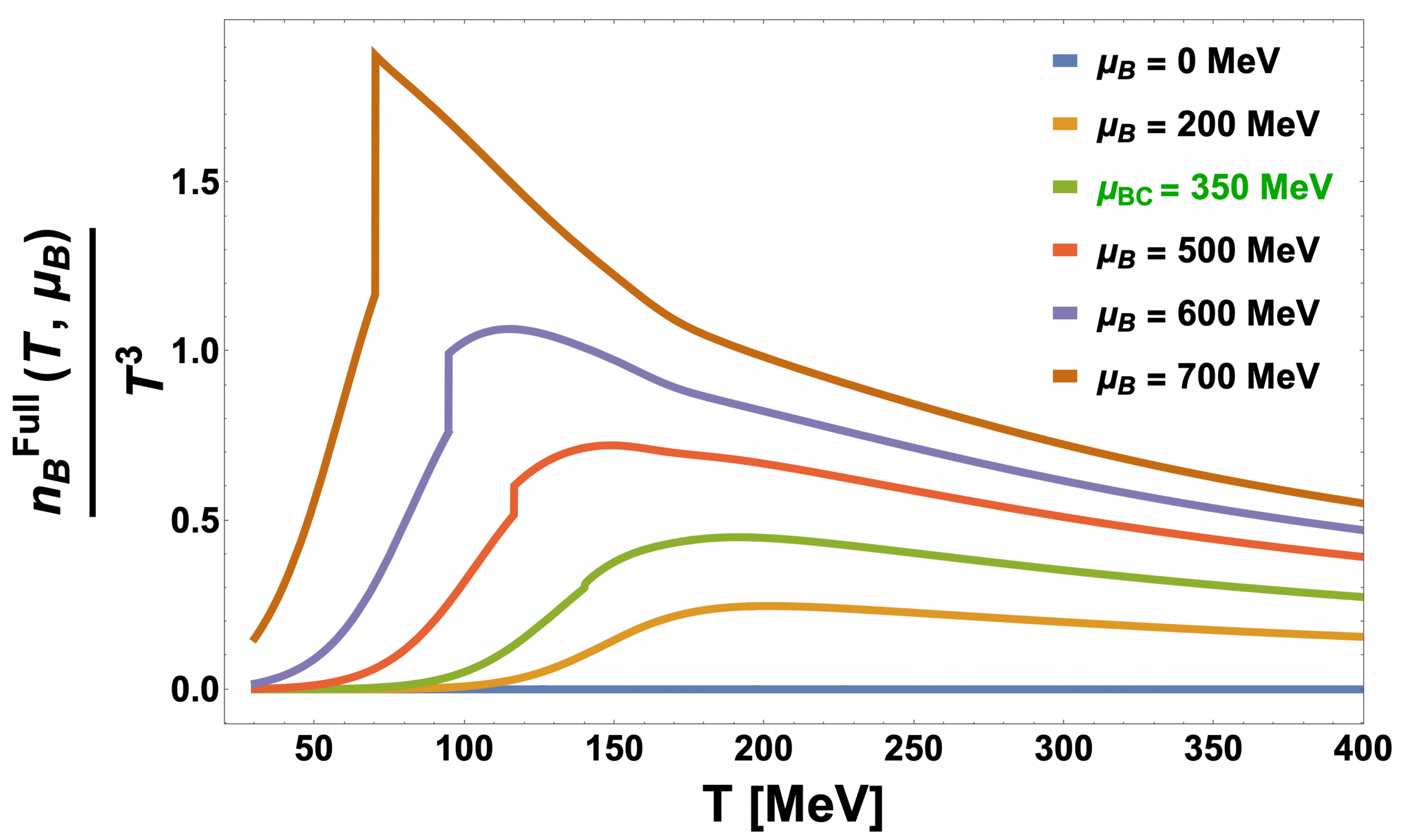
Borsányi, S et al PRL. 108(1), 101.034901(2021)]

Baryon density results

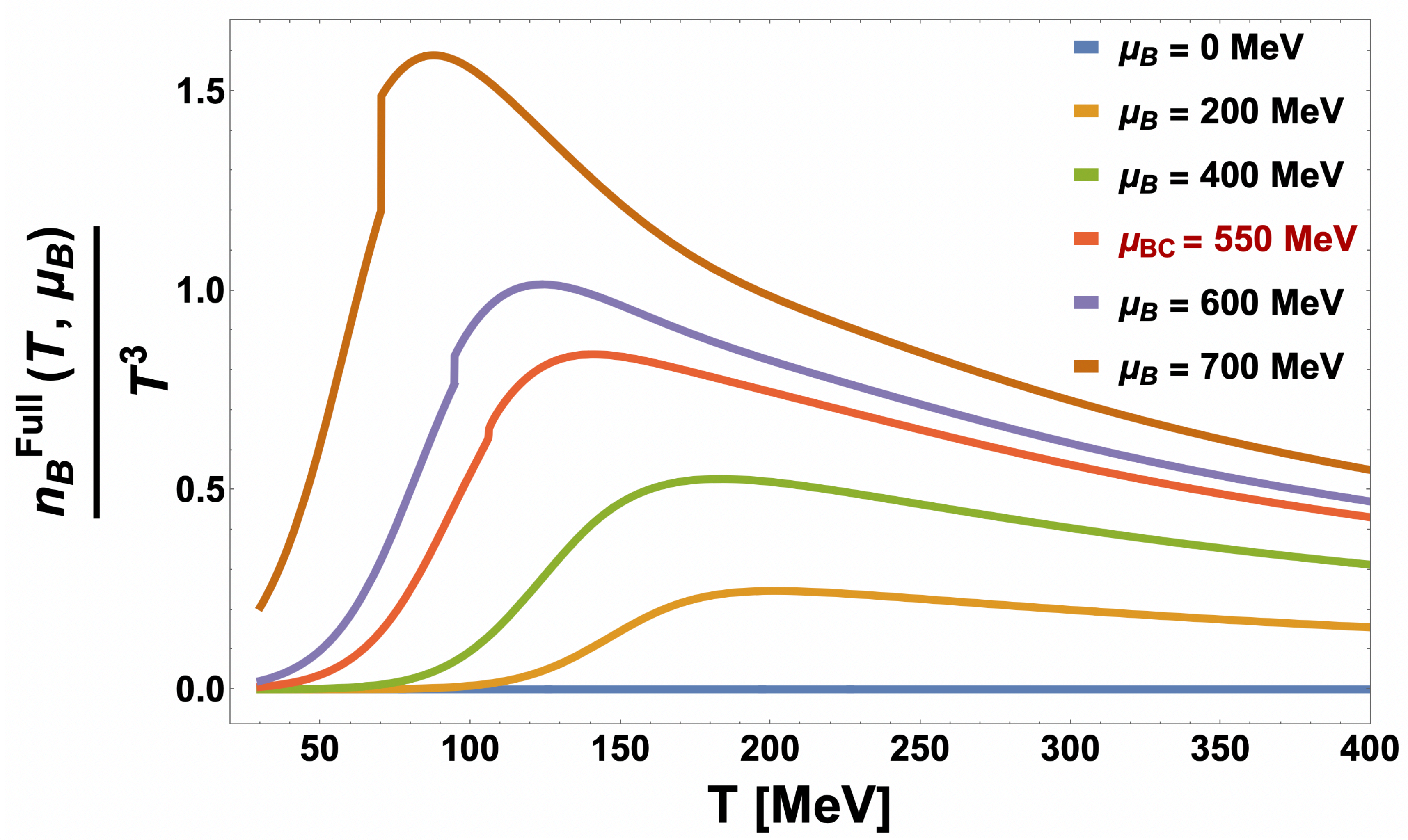
Baryon Density at a constant μ_B for different μ_{BC}

$\alpha_{12} = 90, \quad \rho = 2, \quad w = 2$

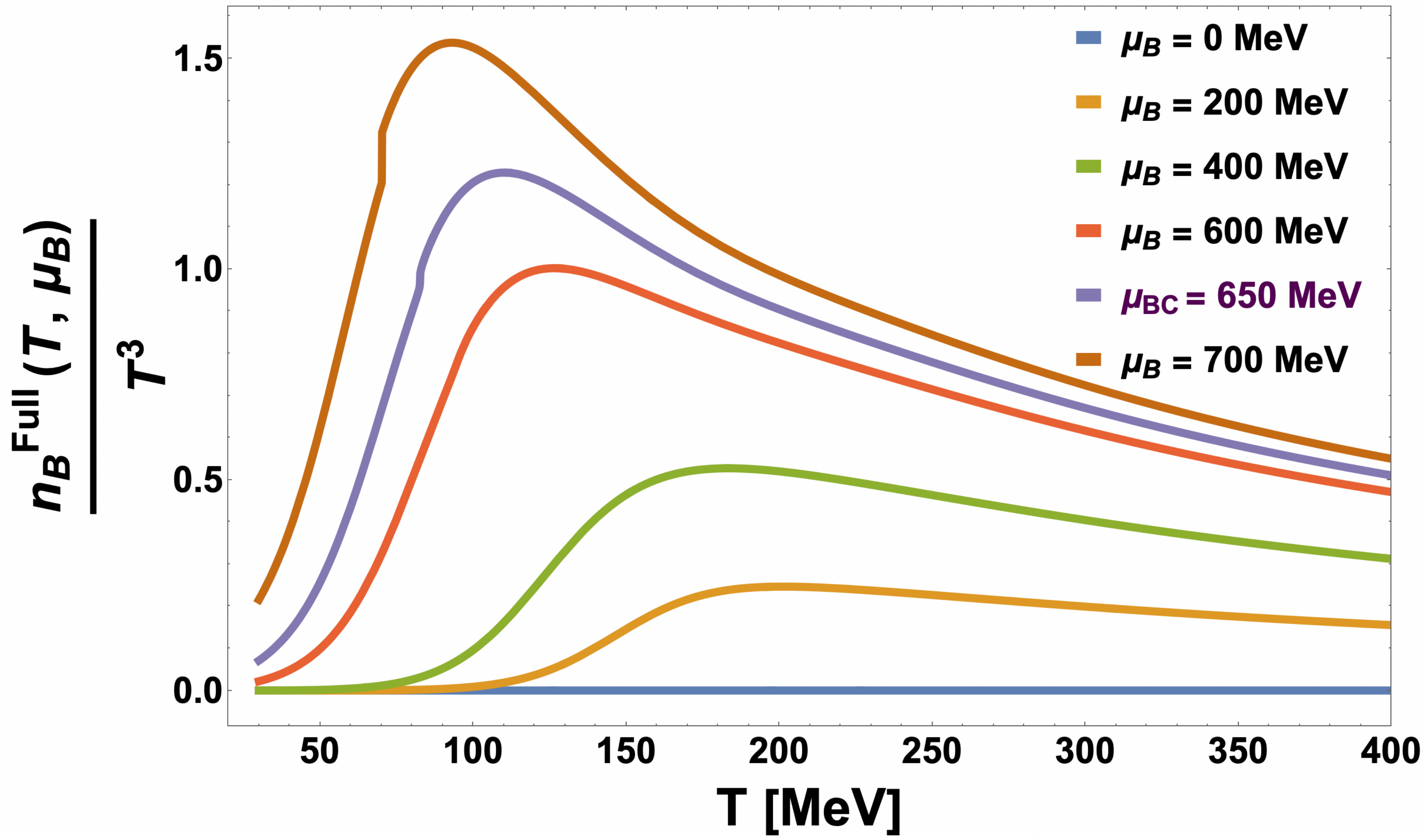
$\mu_{BC} = 350 \quad MeV$



$\mu_{BC} = 550 \quad MeV$



$\mu_{BC} = 650 \quad MeV$

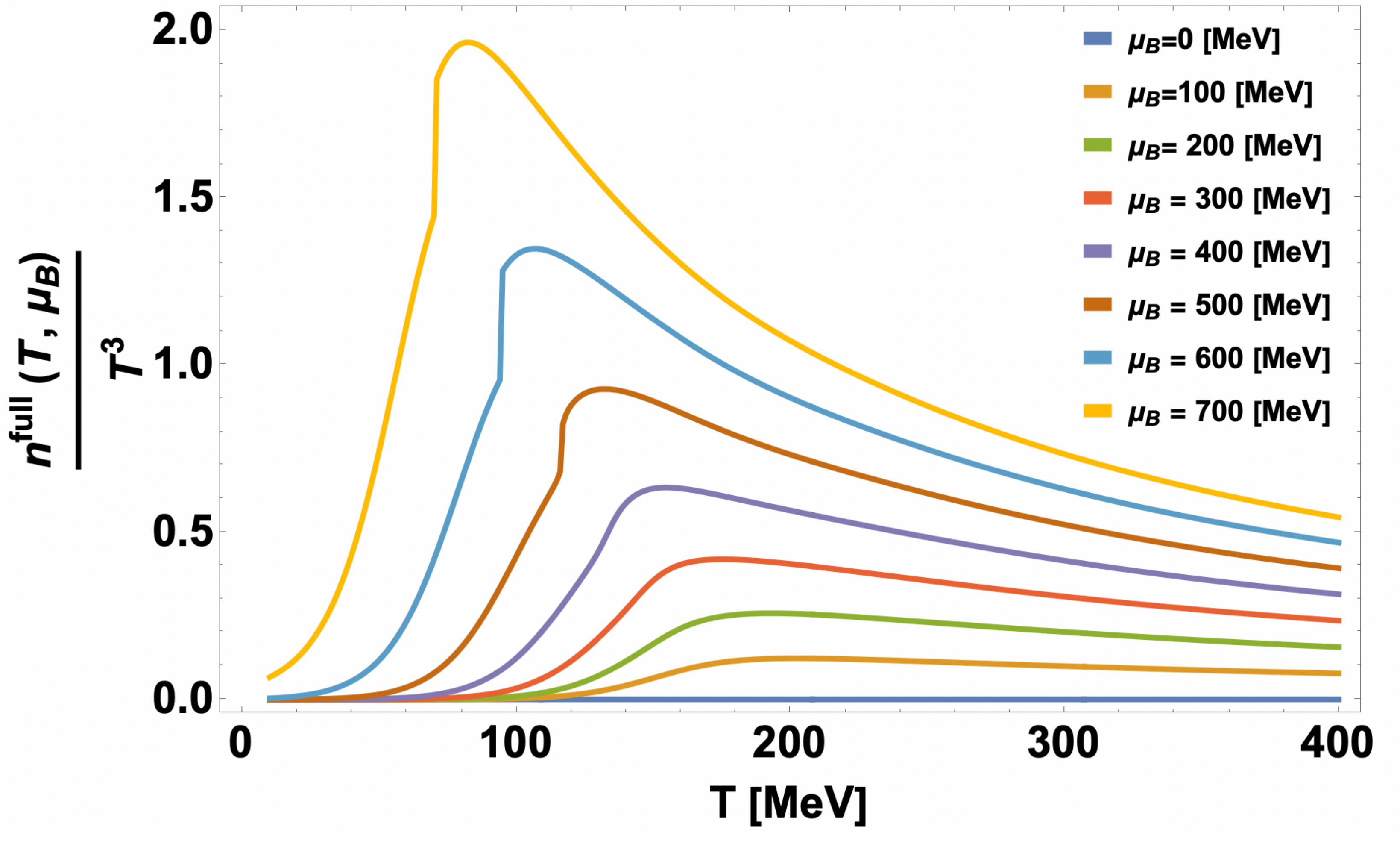


Estimating the Critical contribution

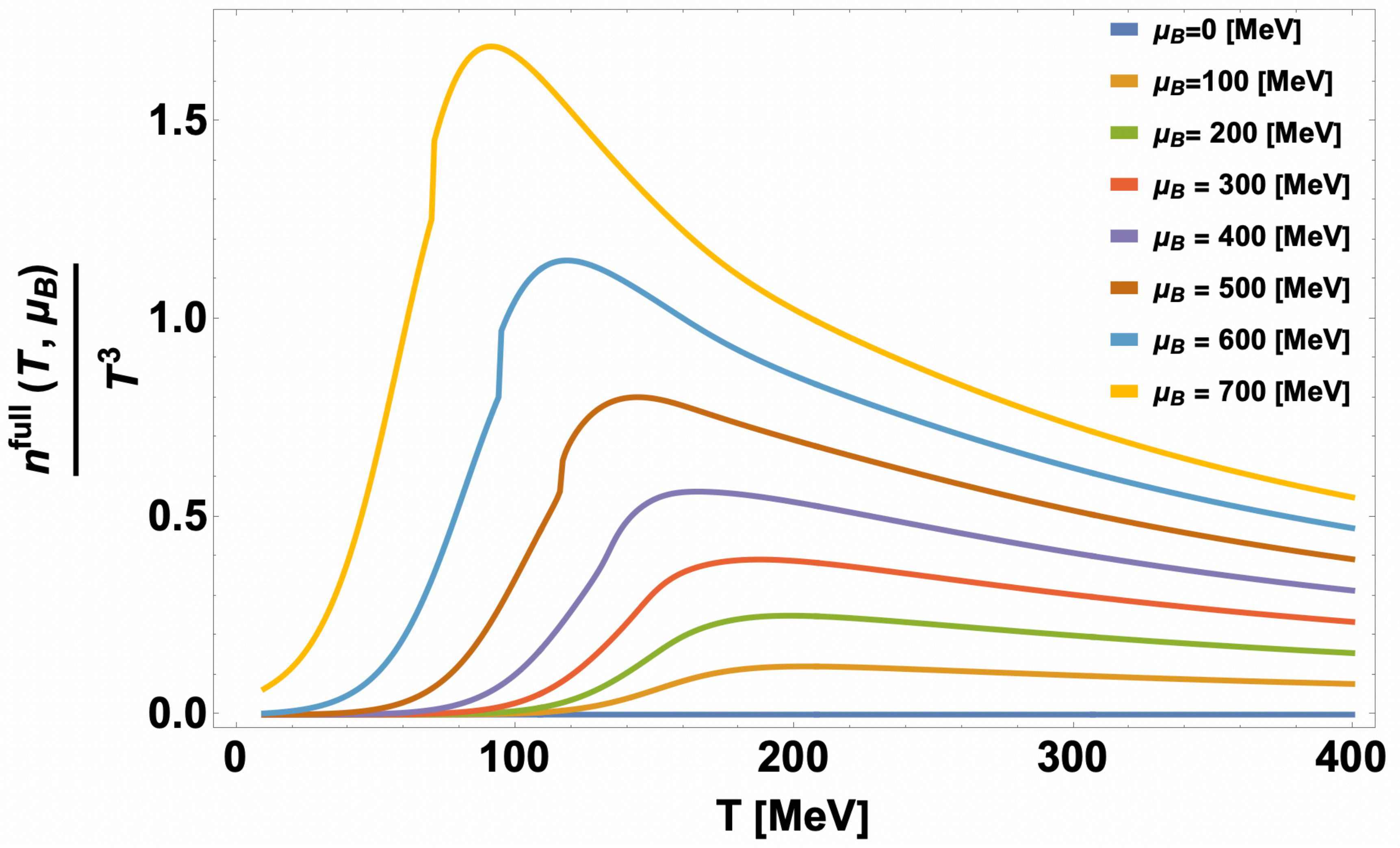
Baryon Density at a constant μ_B for w=1 to w=10

$\mu_{BC} = 500\ [\text{MeV}],\ \ \alpha_{12} = 90,\ \ \rho = 20$

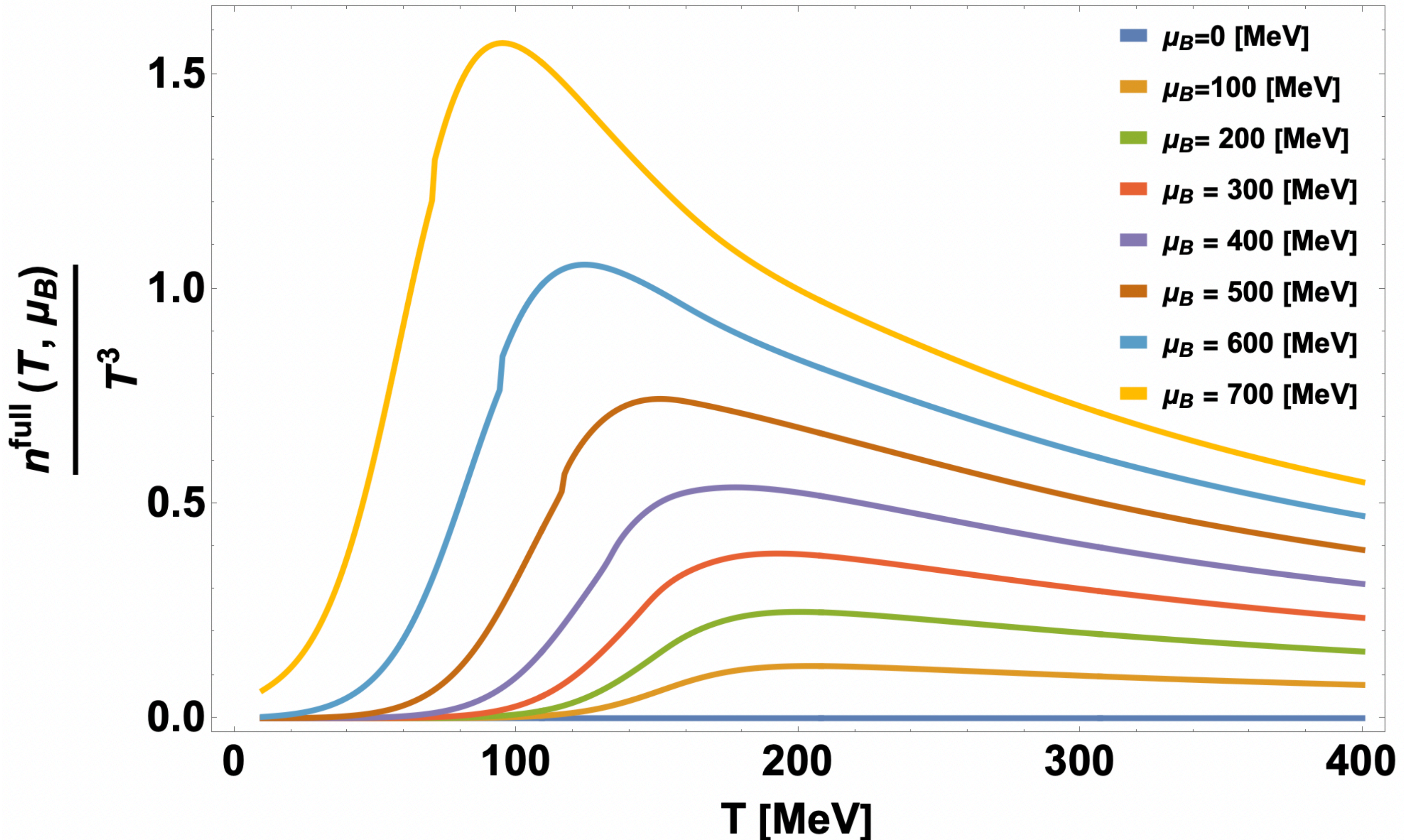
$w = 0.5$



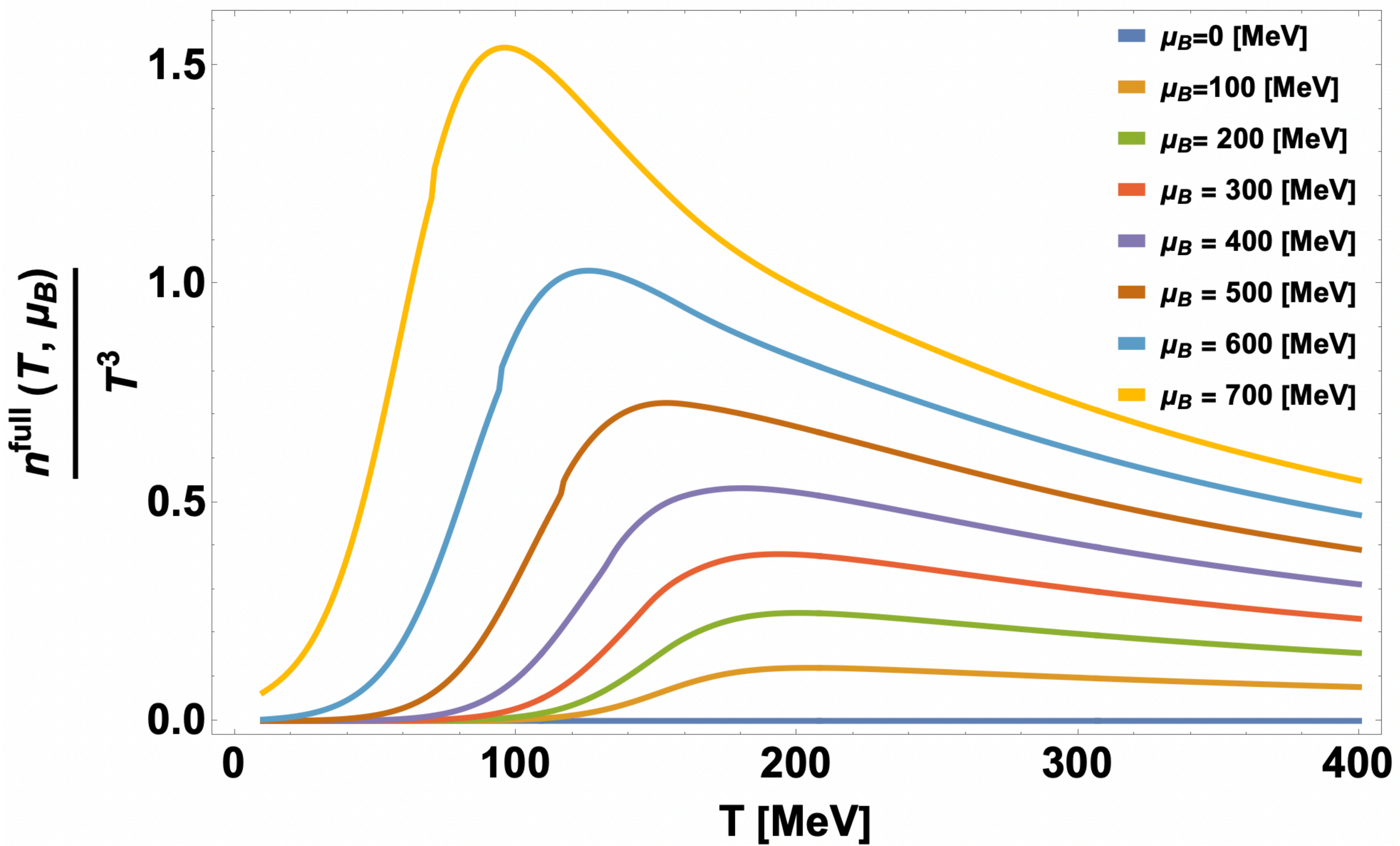
$w = 1$



$w = 2$



$w = 3$



Thermodynamic Relations

$$\frac{P(T,\mu_B)}{T^4} = \frac{P(T,0)}{T^4} + \int_0^{\mu_B} d\hat{\mu}'_B \frac{n_B(T,\hat{\mu}'_B)}{T^3}$$

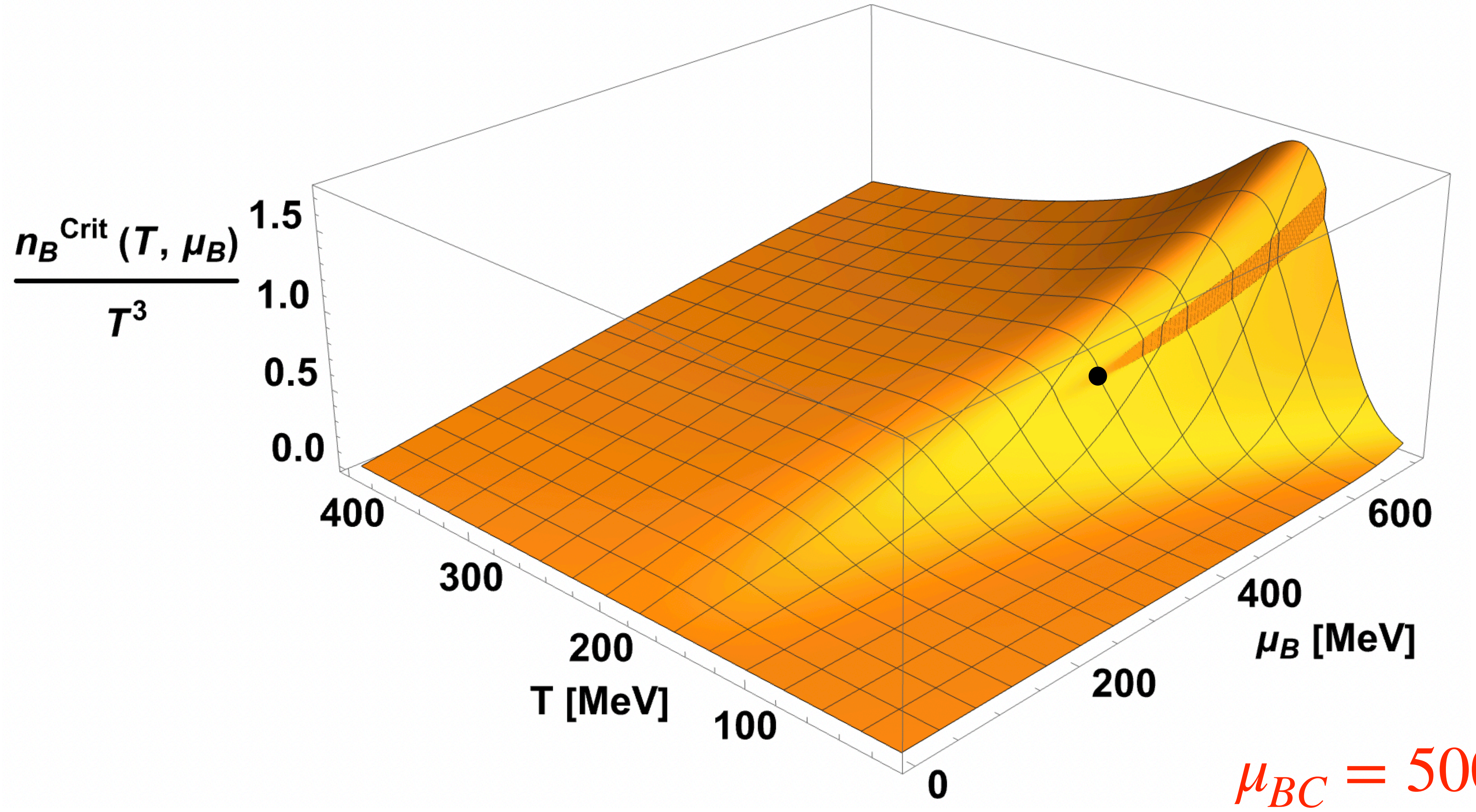
$$\frac{s(T,\mu_B)}{T^3} = \frac{1}{T^3}\left(\frac{\partial P}{\partial T}\right)\Bigg|_{\mu_B}$$

$$\frac{\epsilon(T,\mu_B)}{T^3} = \frac{s}{T^3} - \frac{P}{T^4} + \frac{\mu_B}{T}\frac{n_B}{T^3}$$

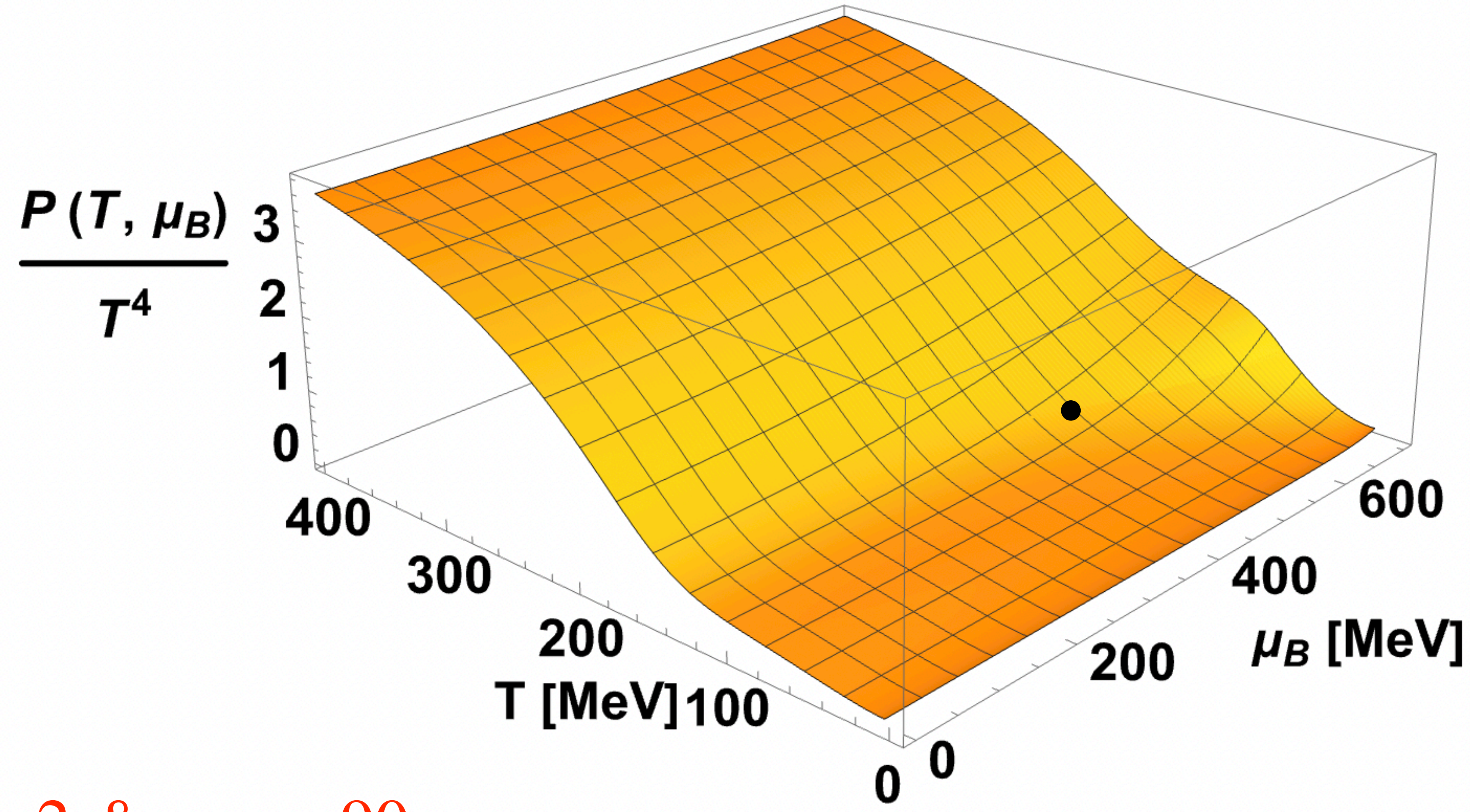
$$\frac{\chi_2(T,\mu_B)}{T^2} = T\left(\frac{\partial(n_B/T^3)}{\partial \mu_B}\right)\Bigg|_T$$

Thermodynamic Observables

Baryon Density



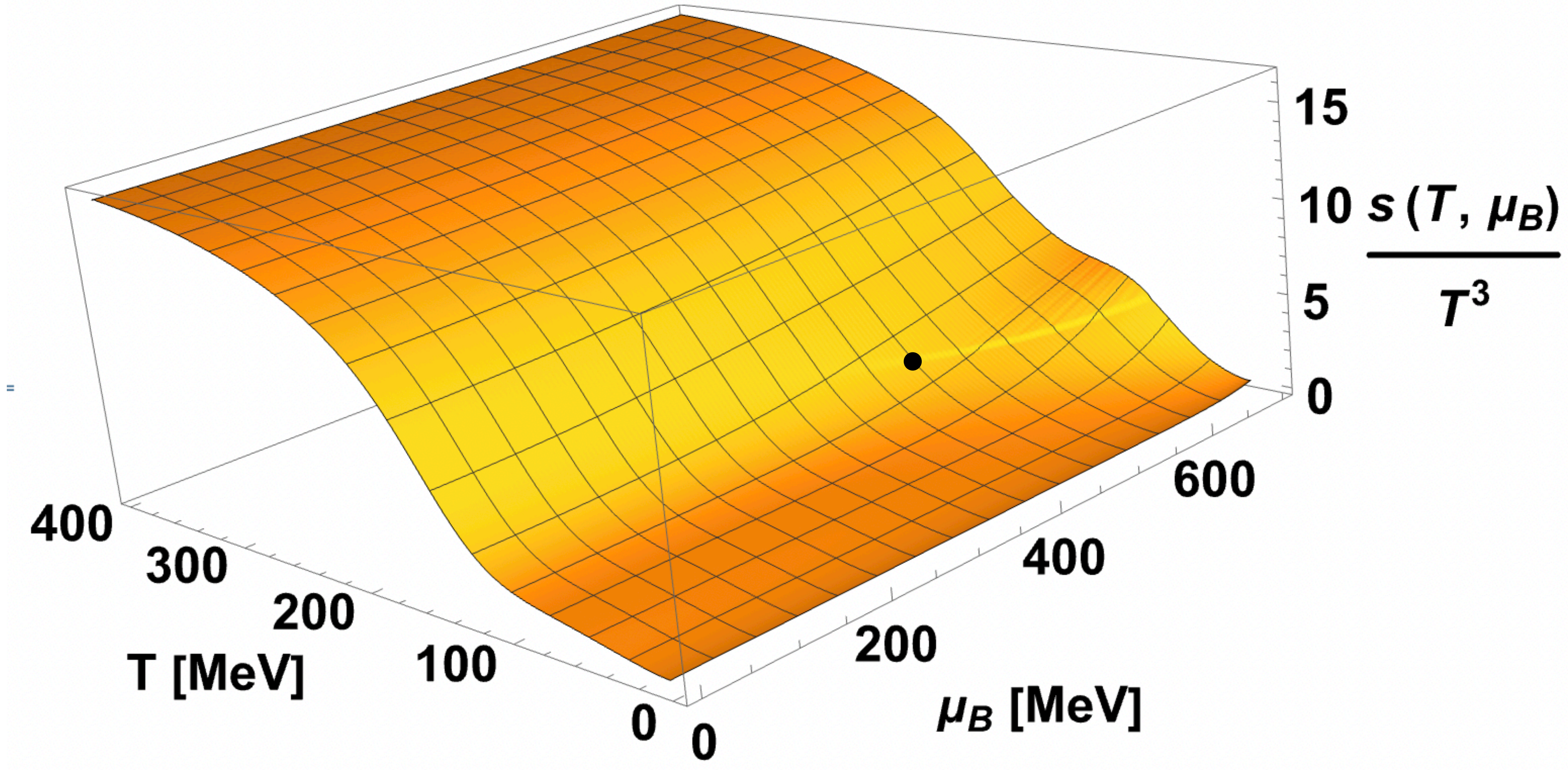
Pressure



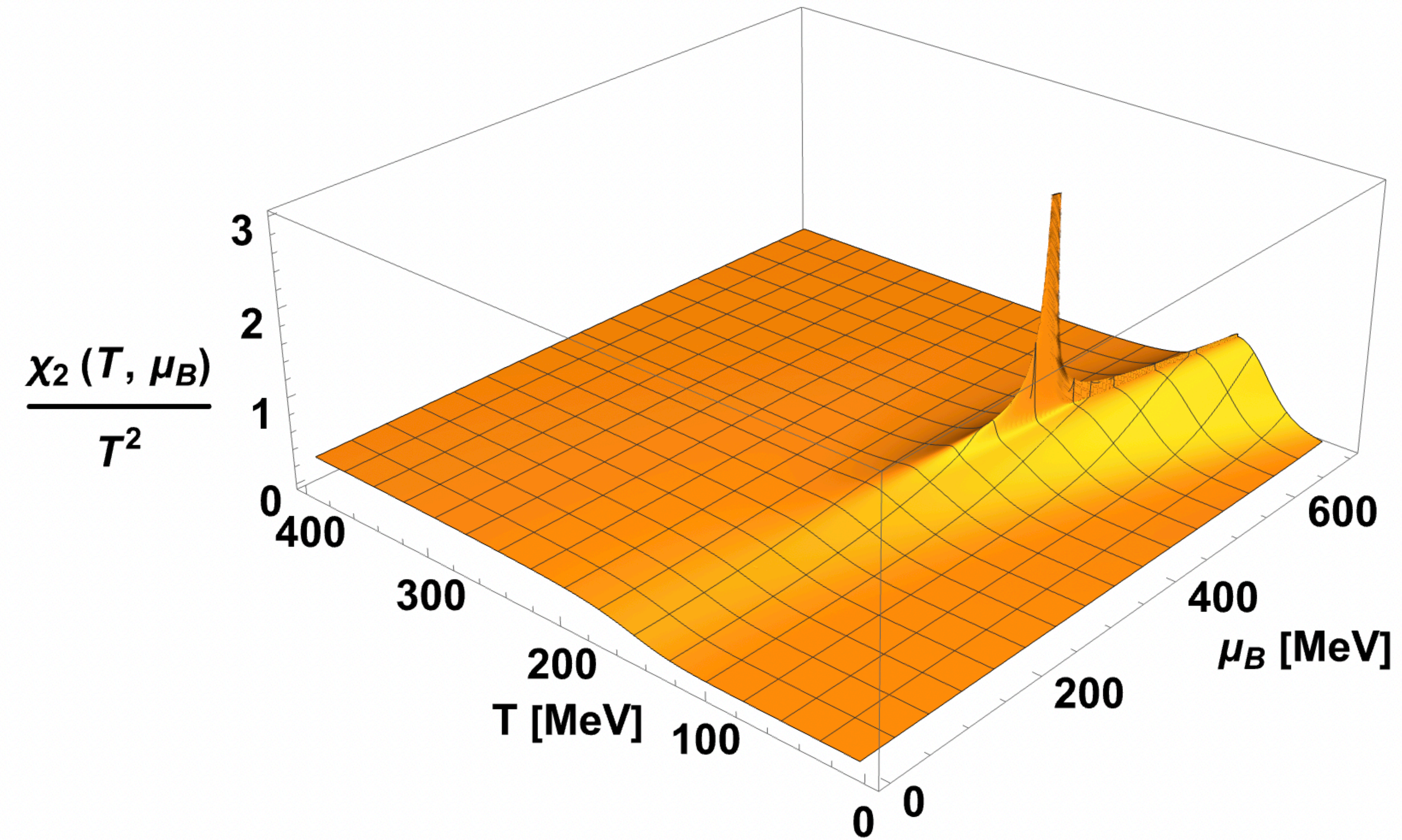
● **Critical Point**

$\mu_{BC} = 500$ **MeV** $w = 20$, $\rho = 2$ & $\alpha_{12} = 90$

Entropy Density

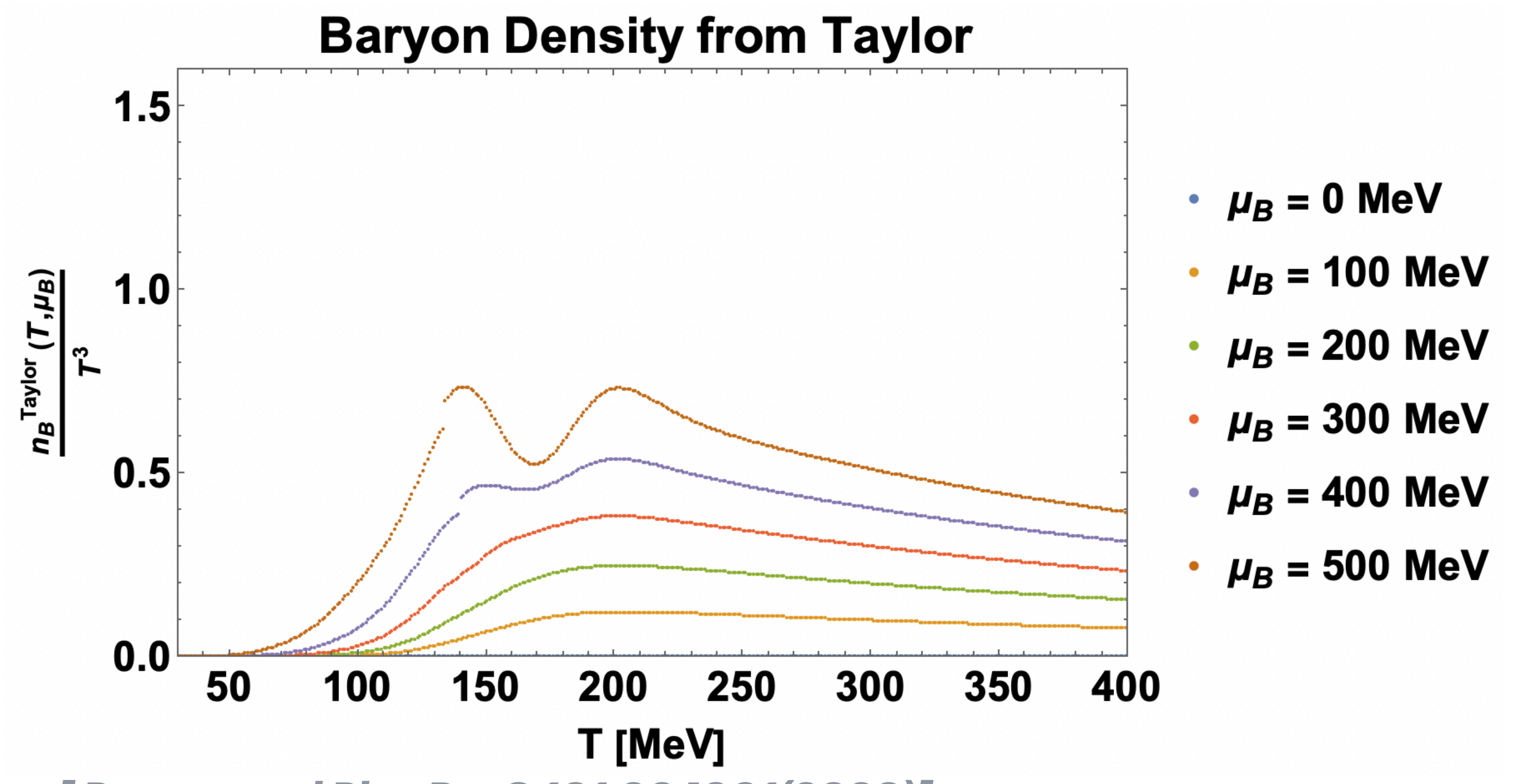


Baryon number Susceptibility

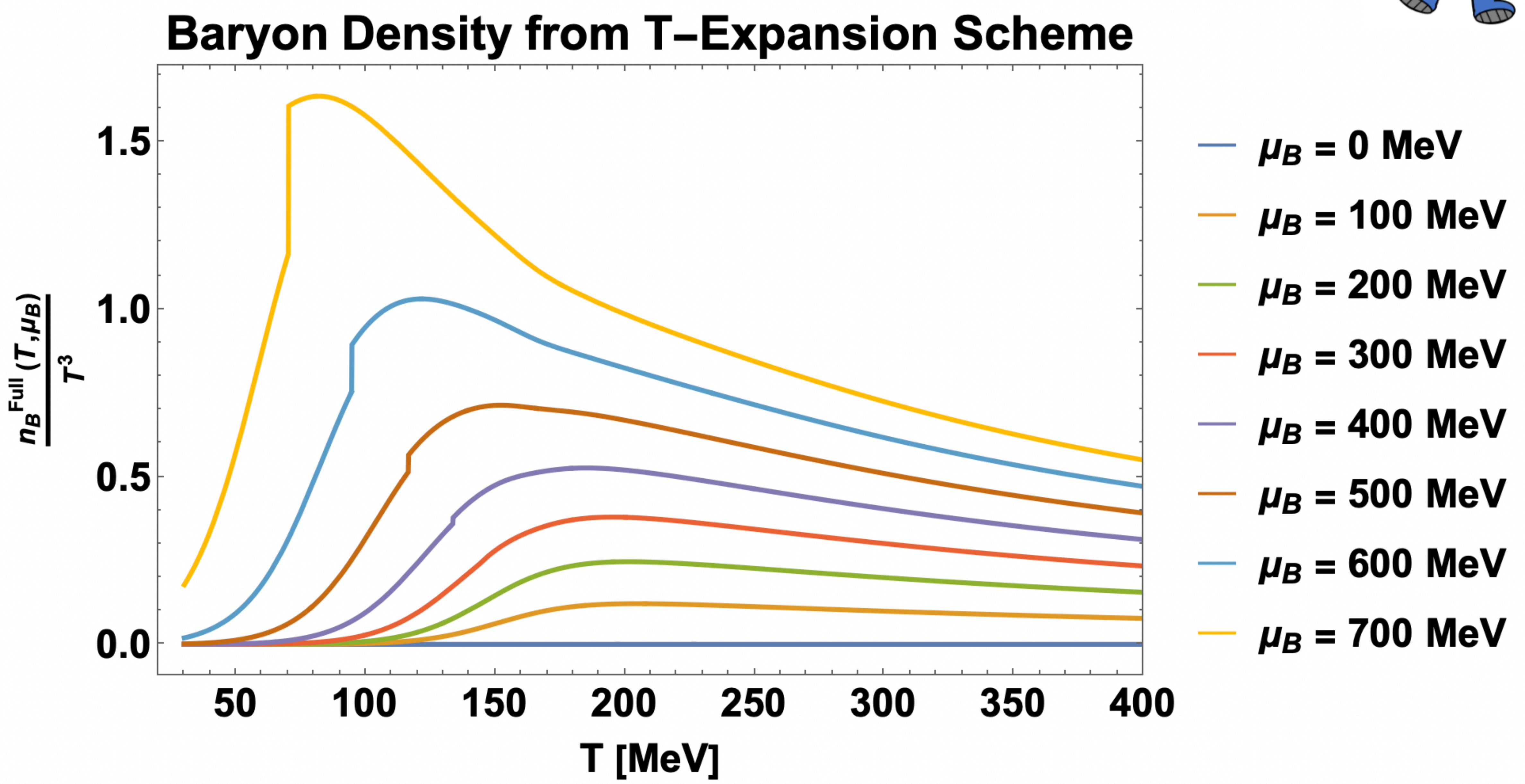


Summary

$\mu_B = 350$ [MeV], $\alpha_{12} = 90$, w $= 3$, $\rho = 2$



[Parotto et al PhysRevC.101.034901(2020)]



Disclaimer! : We don't predict the location of the critical point

- **We provide a family of EoS with a 3D Ising critical point up to $\mu_B = 700\ MeV$ and match lattice at low μ_B.**
- **Our EoS allows users to change parameters and use it as input in hydrodynamical simulations.**

Outlook

Constrains on the Parameter space

$$\frac{T' - T_0}{T_0} = - w\, h \sin \alpha_{12}$$

$$\frac{\mu_B^2 - \mu_{BC}^2}{T_0^2} = w(-r\rho - h\cos \alpha_{12})$$

- Stability $\quad c_v = \left(\frac{\partial s}{\partial T}\right)\Big|_{n_B} > 0 \qquad \chi_T(T, \mu_B) = \left(\frac{\partial n_B}{\partial \mu_B}\right)\Big|_T = \left(\frac{\partial^2 P}{\partial \mu_B^2}\right)\Big|_T > 0$
- Causality $\quad 0 < c_s^2(T, \mu_B) < 1$

[Floerchinger, S. et al PhysRevC. 92(6), 064906.(2015)]

[Mroczek, D.. et al PhysRevC. 107(5), 054911.(2023)]

Still under investigation

Thank you !

Back up !

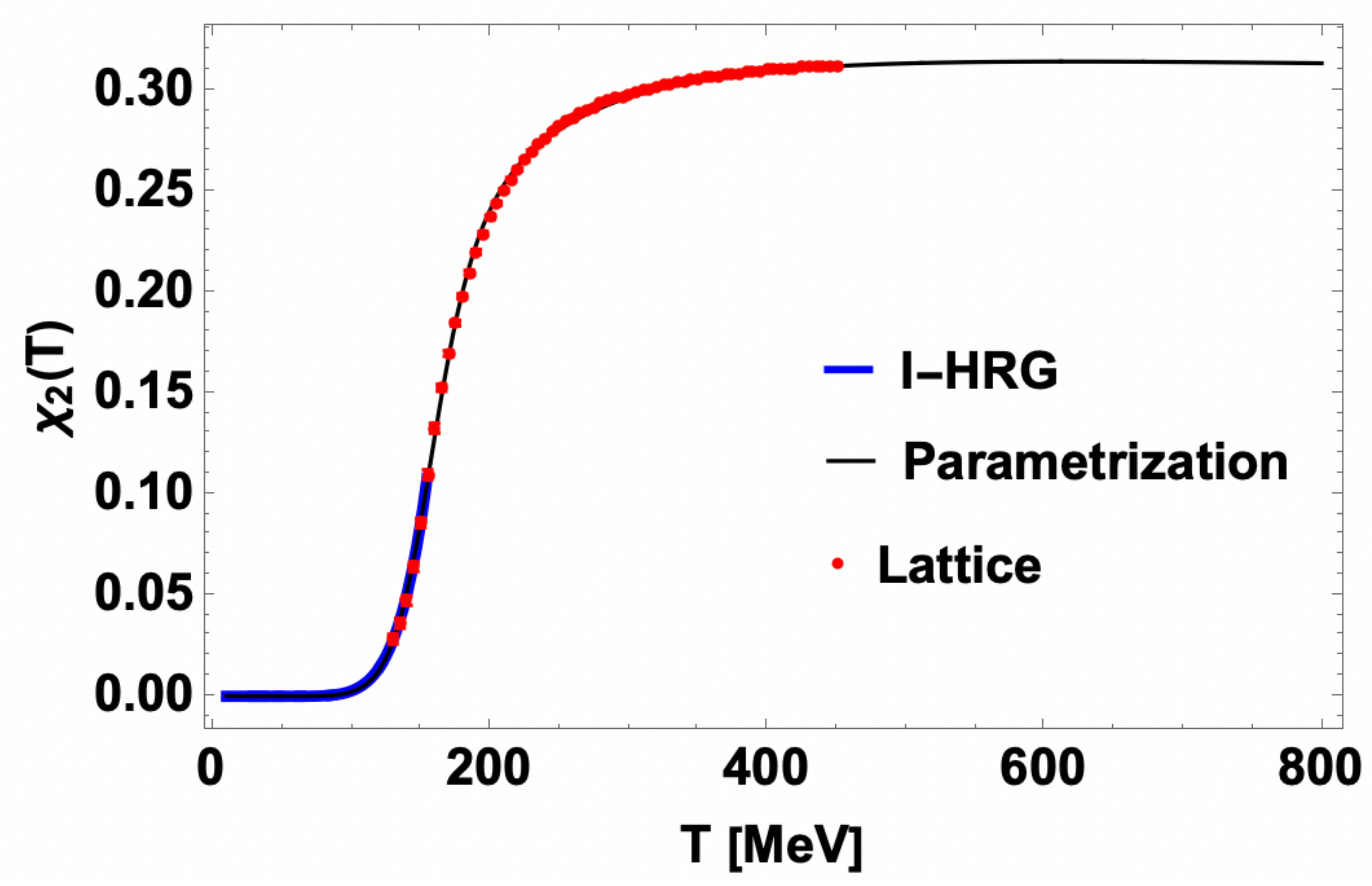
Parametrize Lattice data

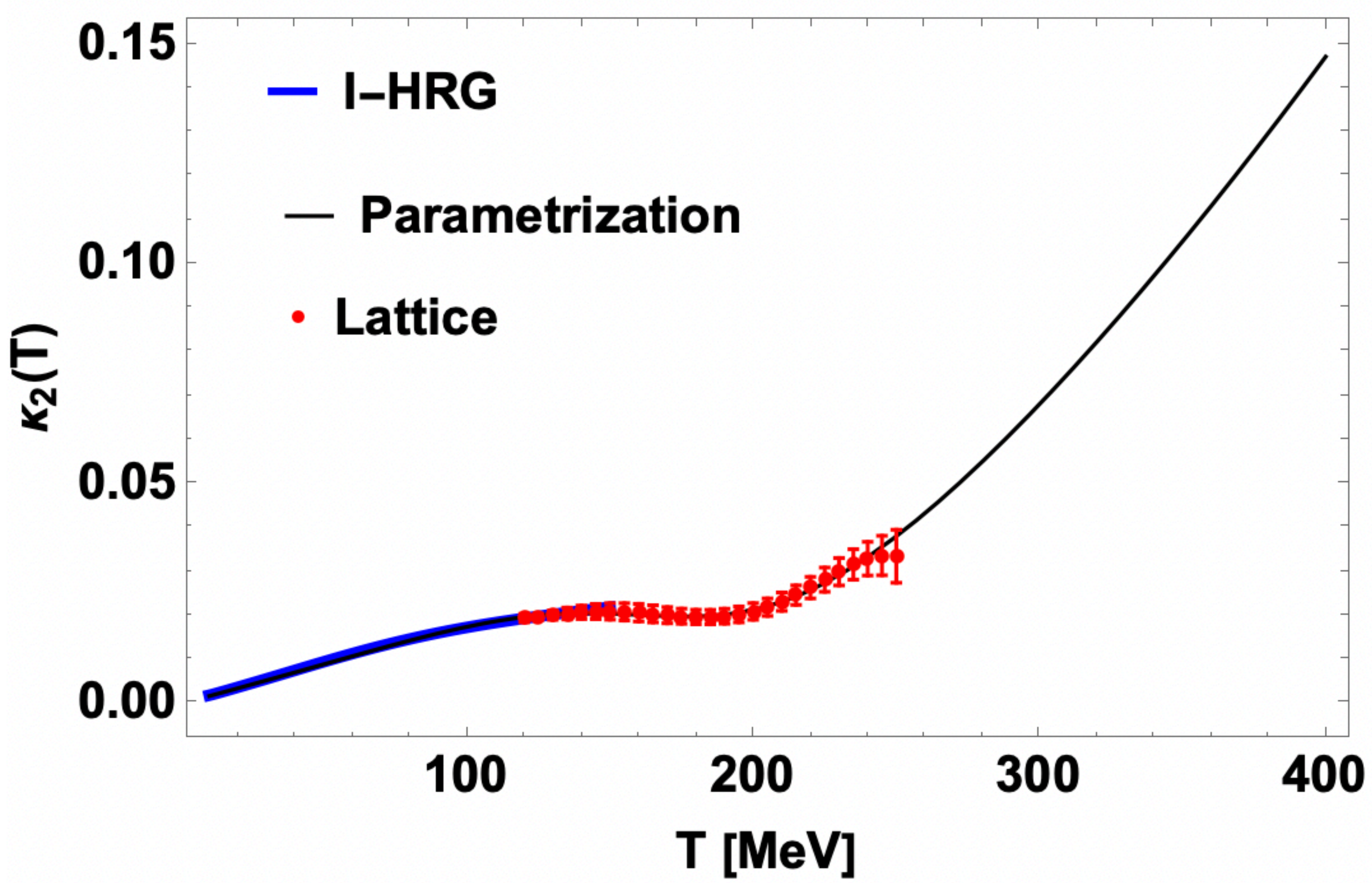
$$\chi^{B}_{2,lattice}(T) = e^{-h_1/x' - h_2/x'^2} . f_3 . (1 + \tanh(f_4 x' + f_5))$$

$$x' = \frac{T}{154}$$

$$\kappa^{B}_{2,lattice}(T) = \frac{a0 + a1x + a2x^2 + a3x^3 + a4x^4 + a5x^5}{b0 + b1x + b2x^2 + b3x^3 + b4x^4 + b5x^5}$$

$$x = \frac{T}{200}$$



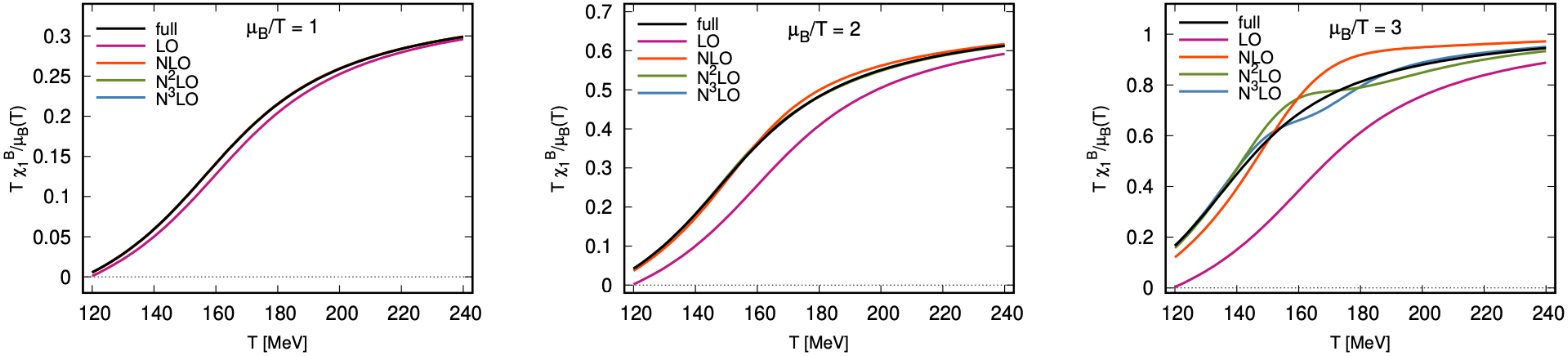


We merge $\chi_{2,lattice}(T)$ and $\kappa_2(T)$ with IHRG, however, We also need to merge all the thermodynamic after introducing the critical point

Taylor Expansion of $\frac{\chi_1^B(T,\hat{\mu}_B)}{\hat{\mu}_B}$

$$\frac{\chi_1^B}{\hat{\mu}_B}(T,\hat{\mu}_B) = \chi_2^B(T,0) + \frac{\hat{\mu}_B^2}{6}\chi_4^B(T,0) + \frac{\hat{\mu}_B^4}{120}\chi_6^B(T,0) + \cdots$$

See how it works with Taylor Expansion!

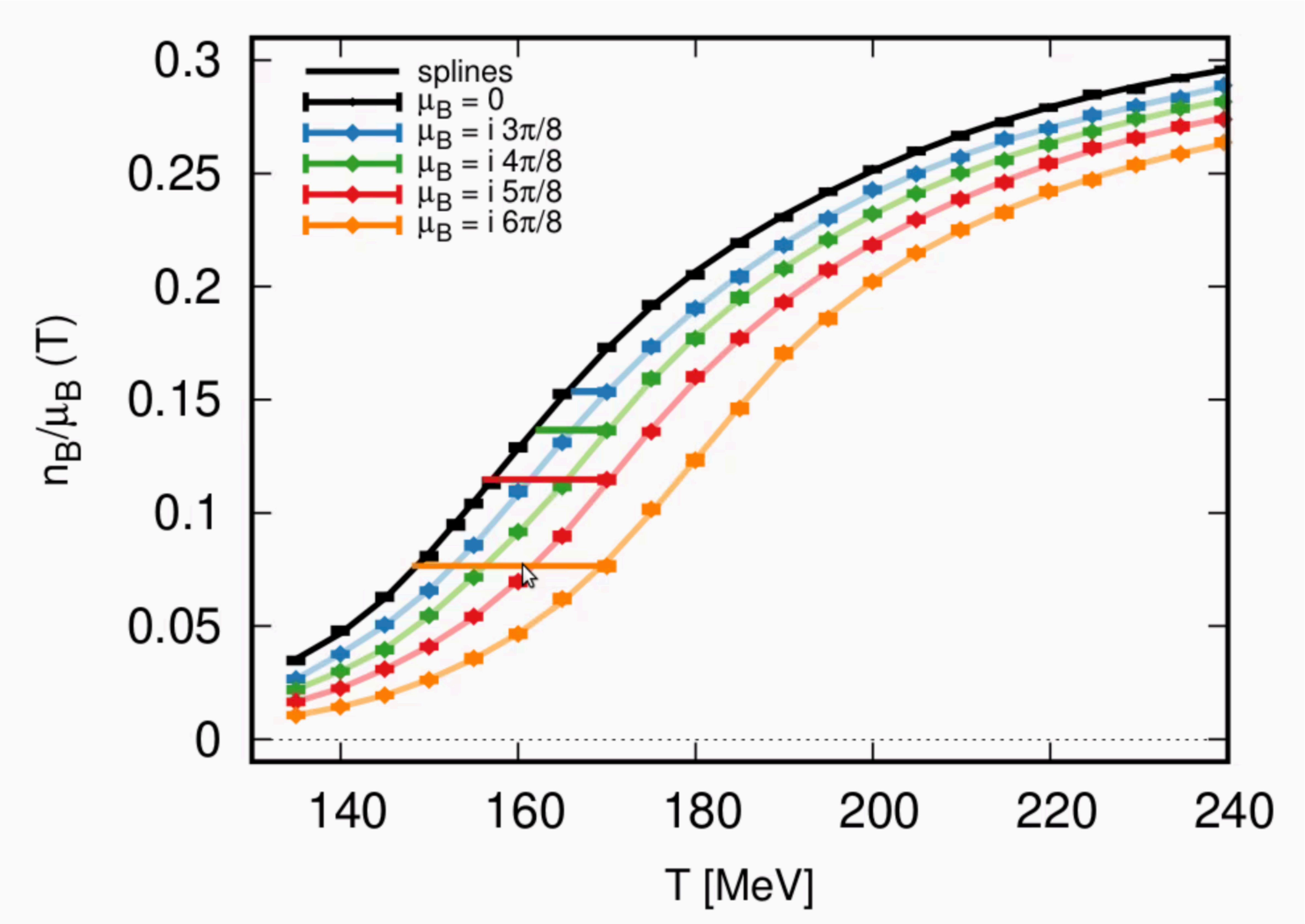


Fluctuations in baryon number and strangeness

$$\frac{\chi_1^S}{\hat{\mu}_B}(T,\hat{\mu}_B) = \chi_{11}^{BS}(T,0) + \frac{\hat{\mu}_B^2}{6}\chi_{31}^{BS}(T,0) + \frac{\hat{\mu}_B^4}{120}\chi_{51}^{BS}(T,0) + \cdots$$

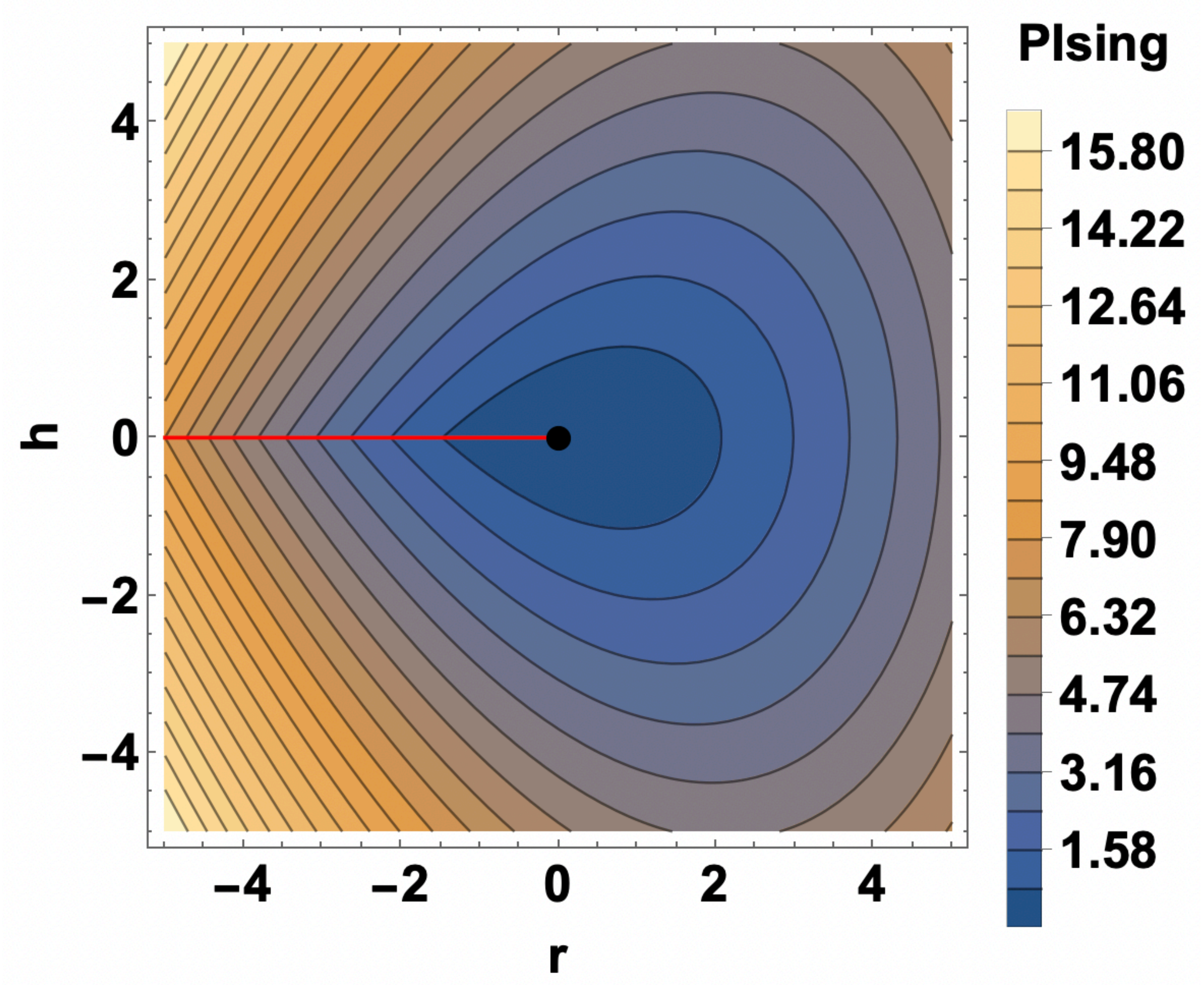
Borsányi, S et al PRL. 108(1), 101.034901(2021)]

Shifting $T\frac{n_B(T,\mu_B)}{\mu_B}$ w/ constant $\kappa_2\left(\frac{\mu_B}{T}\right)^2$

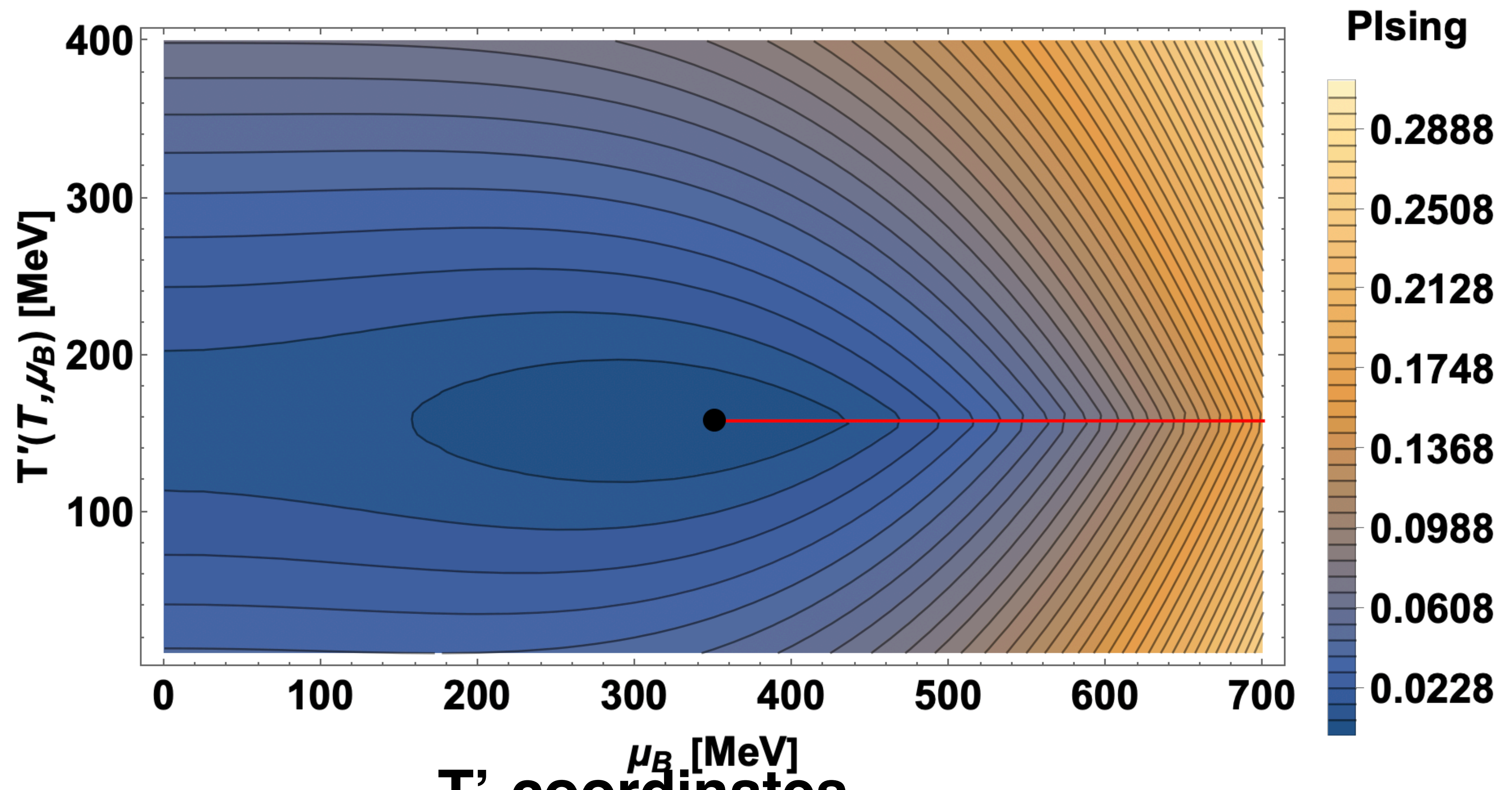


[Parotto P slides et al. PRL

Ising Pressure



3D-Ising coordinates

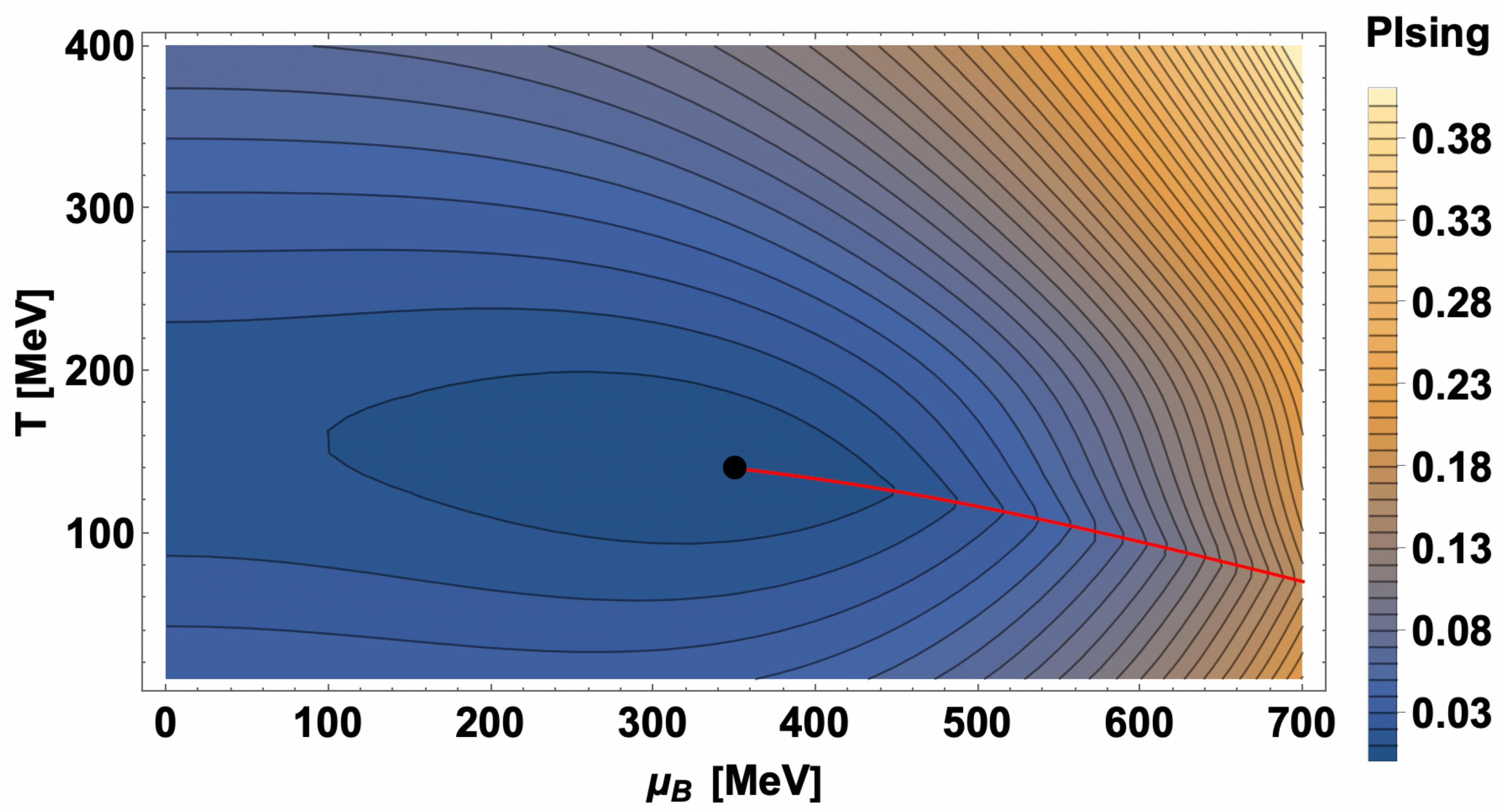


T'-coordinates

Parameters

$$w = 10,\ \rho = 0.5,\ \mu_{BC} = 350\ MeV,\ T_0 = 158\ [MeV]$$

$$T_C\left[1 + \kappa_2(T)\left(\frac{\mu_B}{T_C}\right)^2\right] = T_0$$

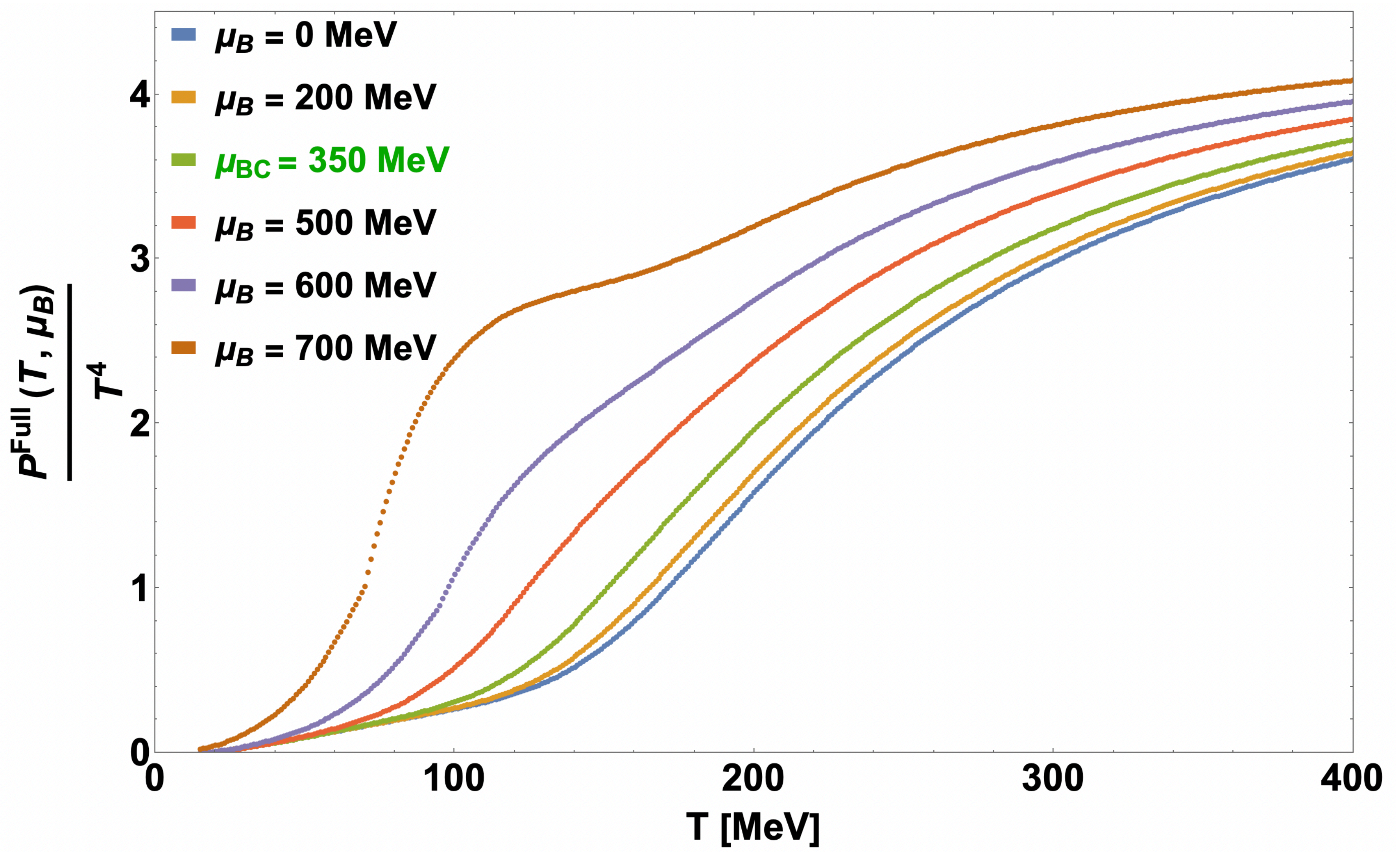


QCD coordinates

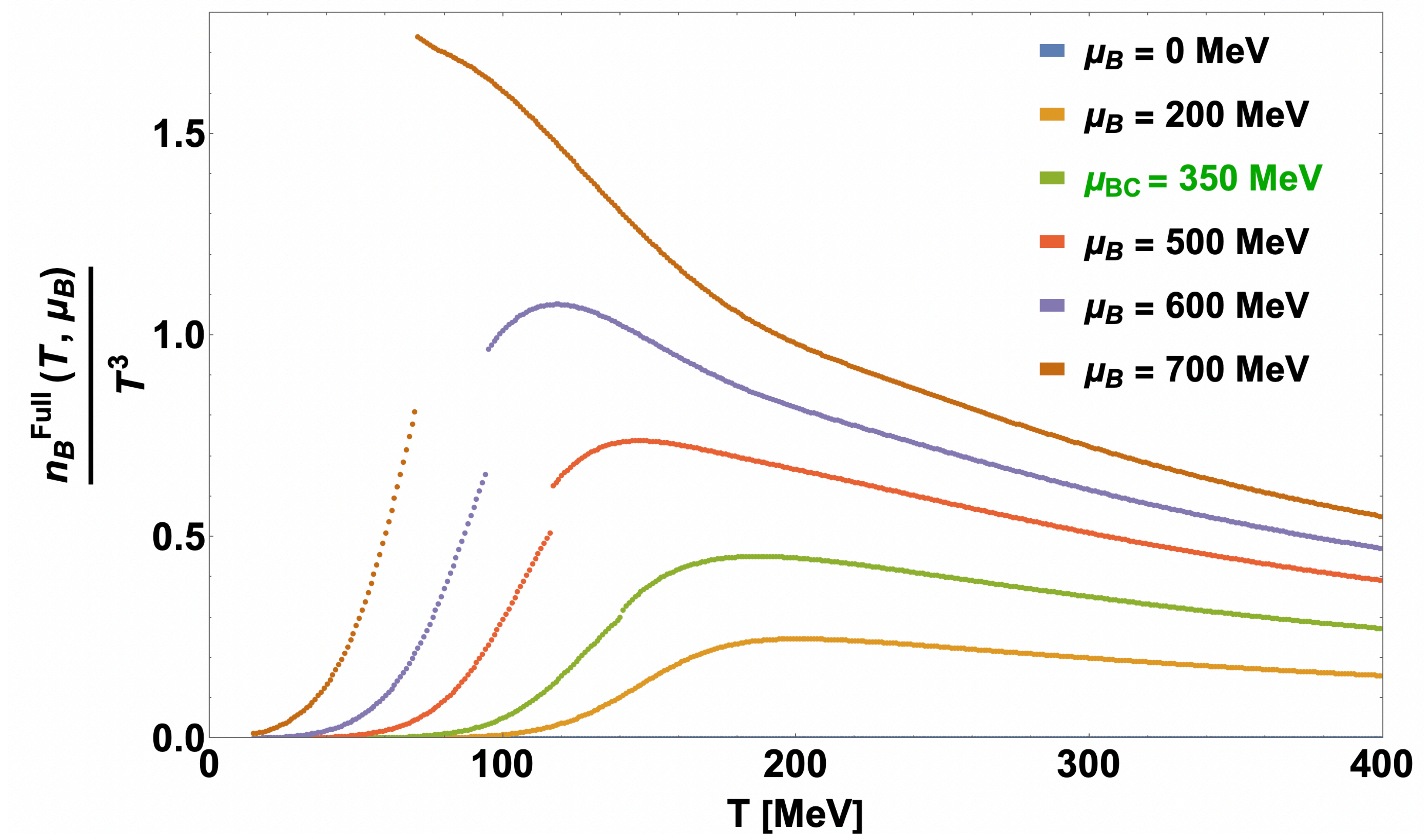
Thermodynamic observables

Ising-AltExS

Pressure



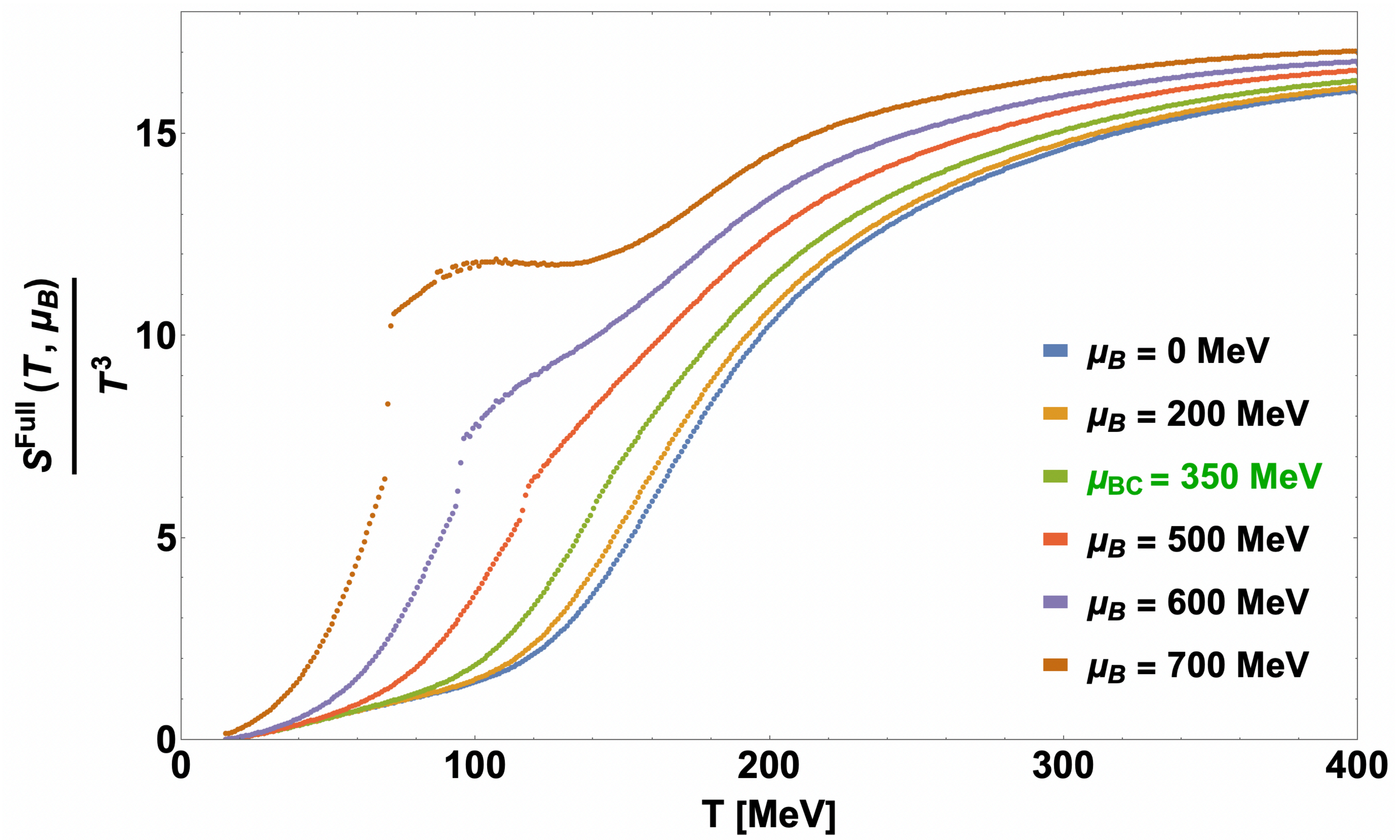
Baryon Density



$\mu_{BC} = 350$ MeV

$T_C = 140$ MeV

Entropy Density



Energy Density